A VARIATIONAL PROBLEM FOR THE ROOTS OF A CUBIC EQUATION*

(A contribution to the theory of servomechanisms)

By HANS BUCKNER (Minden, Germany)

1. The problem. Let \( t \) denote the time, \( u(t) \) a function of time and \( v(t) \) the displacement of a servomechanism, which is to simulate \( u(t) \). As regards the error

\[ \eta(t) = v(t) - u(t) \]

many servomechanisms obey the equation

\[ \eta''' + a_1 \eta'' + a_2 \eta' + a_3 \eta = -u''' - a_4 u'' \] (1)

with constant coefficients \( a_1, \ldots, a_4 \). The quality of the servomechanism depends on the roots of the algebraic equation

\[ x^3 + a_1 x^2 + a_2 x + a_3 = 0. \] (2)

These may be denoted by \( x_1, x_2, x_3 \). We consider the quantity

\[ \xi = \operatorname{Min} \left( -\Re x_1, -\Re x_2, -\Re x_3 \right), \]

where \( \Re x_i \) means the real part of \( x_i \). Then \( \xi > 0 \) means stability, and the greater \( \xi \), the faster the servomechanism follows sudden changes of \( u(t) \). A positive \( \xi \) appears if and only if the well known Hurwitz-conditions are satisfied. These are

\[ a_1 > 0, \quad a_2 > 0, \quad a_1 a_2 > a_3 > 0. \]

If \( a_1 \) and \( a_2 \) are regarded as fixed, and \( a_3 \) as variable, \( \xi \) is a function of \( a_3 \). An interesting problem both from a mathematical and a technical point of view, is to find \( \operatorname{Max} \xi \). There is a simple formula for it, which does not seem to be as well known as its merits.

2. The solution. Introducing the classic notations

\[ p = a_2 - \frac{a_1^2}{3}, \quad q = a_3 - \frac{a_1 a_2}{3} + \frac{2}{27} a_1^3, \quad D = -4p^3 - 27q^2, \]

we may represent the roots by

\[ x_1 = -\frac{a_1}{3} + \epsilon \left[ -\frac{q}{2} + \frac{1}{18} (-3D)^{1/2} \right]^{1/3} + \epsilon^{2} \left[ -\frac{q}{2} - \frac{1}{18} (-3D)^{1/2} \right]^{1/3}, \]

\[ x_2 = -\frac{a_1}{3} + \epsilon^{2} \left[ -\frac{q}{2} + \frac{1}{18} (-3D)^{1/2} \right]^{1/3} + \epsilon \left[ -\frac{q}{2} - \frac{1}{18} (-3D)^{1/2} \right]^{1/3}, \]

\[ x_3 = -\frac{a_1}{3} + \left[ -\frac{q}{2} + \frac{1}{18} (-3D)^{1/2} \right]^{1/3} + \left[ -\frac{q}{2} - \frac{1}{18} (-3D)^{1/2} \right]^{1/3}, \] (3)

*Received March 28, 1949.
where
\[ \epsilon = -\frac{1}{2} + i \frac{1}{2} (3)^{1/2}. \]

The quantity \( p \) depends on \( a_1 \) and \( a_2 \) only.

The roots \( x_1, x_2, x_3 \) are continuous functions of \( a_3 \). Hence the same holds for \( \xi \).

The quantity \( \xi \) vanishes for \( a_3 = 0 \) and for \( a_3 = a_1 a_2 \). In the first case one root vanishes, in the latter \( x = \pm i (a_2)^{1/2} \) represents two roots with vanishing real parts.

For \( 0 < a_3 < a_1 a_2 \) we have \( \xi > 0 \). Therefore \( \xi \) as a function of \( a_3 \) has a positive maximum in the interval \((0, a_1 a_2)\).

The criterion for multiple roots is \( D = 0 \). For \( D > 0 \) three different real roots exist; for \( D < 0 \) one pair of conjugate complex roots exists. The roots \( x_1 \) and \( x_2 \) of the system (3) give these complex roots, while \( x_3 \) is real.

The only case where \( D \) may vanish is for \( p \leq 0 \). Vanishing takes place for two values of \( a_3 \) which we denote by \( (\alpha - \beta) \) and \( (\alpha + \beta) \). We can write

\[ \alpha = \frac{a_1 a_2}{3} - \frac{2a_1}{27}, \quad \beta = 2 \left( -\frac{p^{1/2}}{27} \right). \]

All roots are continuously differentiable functions of \( a_3 \) in all intervals which do not contain \( (\alpha - \beta) \) or \( (\alpha + \beta) \). Starting from the well-known formulae

\[
\begin{align*}
x_1 + x_2 + x_3 &= -a_1, \\
x_1^2 + x_2^2 + x_3^2 &= a_1^2 - 2a_2, \\
x_1^3 + x_2^3 + x_3^3 &= -a_1^3 + 3a_1a_2 - 3a_3,
\end{align*}
\]

we find

\[
\begin{align*}
\frac{dx_1}{da_3} &= \frac{1}{(x_1 - x_2)(x_1 - x_3)}, \\
\frac{dx_2}{da_3} &= \frac{1}{(x_2 - x_1)(x_2 - x_3)}, \\
\frac{dx_3}{da_3} &= \frac{1}{(x_3 - x_1)(x_3 - x_2)}.
\end{align*}
\]

Now, let \( a_3 \) be a value for which \( D > 0 \). Then all roots are real and different from one another. Hence \( -\xi \) coincides with one root, say \( x_i \), for a sufficient small neighbourhood of \( a_3 \). From (4) it follows that

\[ \frac{d\xi}{da_3} = -\frac{dx_i}{da_3} \neq 0, \]

in this neighbourhood. Thus, a maximum of \( \xi \) can exist within the interval \((0, a_1 a_2)\) only if \( D \leq 0 \).

Next, we consider a value \( a_3 \) and a neighbourhood of it in which \( D < 0 \). In this case we may write

\[ \Re x_1 = \Re x_2 = -\frac{a_1}{3} - \frac{1}{2} \left[ -\frac{q}{2} + \frac{1}{18} (-3D)^{1/2} \right]^{1/3} - \frac{1}{2} \left[ -\frac{q}{2} - \frac{1}{18} (-3D)^{1/2} \right]^{1/3} \]
and, furthermore

\[ \xi = \text{Min} (-Rx_1, -x_3). \]

The criterion for

\[ x_3 = Rx_1 \]

is \( q = 0 \). This means that \( a_3 = \alpha \).

If it is assumed that \( a_3 \) does not coincide with \( \alpha \), \( \xi \) is represented either by \( -x_3 \) or by \( -Rx_1 \) for a sufficient small neighbourhood of \( a_3 \). From (4) it follows that

\[ \frac{d\xi}{da_3} = -R \frac{dx_i}{da_3} \neq 0 \]

with a suitable subscript \( i \). This holds for the neighbourhood. Therefore a maximum does not exist if \( D < 0 \) and \( q \neq 0 \).

Summing up, we may state that the maximum of \( \xi \) is given either by \( D = 0 \) or by \( q = 0, D < 0 \).

The following three cases may occur.

1) \( p > 0 \): We find \( D < 0 \). Max \( \xi \) is determined by \( q = 0 \). We have

\[ \text{Max } \xi = \frac{a_1}{3}; \quad a_3 = \frac{a_1a_2}{3} - \frac{2}{27} a_1^3. \]

2) \( p = 0 \): We find \( D = 0, q = 0 \) for \( a_3 = \alpha \) and

\[ \text{Max } \xi = \frac{a_1}{3} \quad \text{for} \quad a_3 = \frac{a_1a_2}{3} - \frac{2}{27} a_1^3. \]

3) \( p < 0 \): It is obvious that \( D > 0 \) for \( q = 0 \). The maximum of \( \xi \) is obtained for \( D = 0 \). The roots then are given by

\[ -x_1 = -x_2 = \frac{a_1}{3} - \left( \frac{q}{2} \right)^{1/3}, \]

\[ -x_3 = \frac{a_1}{3} + 2 \left( \frac{q}{2} \right)^{1/3}. \]

For \( a_3 = \alpha \pm \beta \), we have \( q = \pm \beta \) and

\[ \xi = \frac{a_1}{3} - \left( \frac{\beta}{2} \right)^{1/3} \quad \text{for} \quad a_3 = \alpha + \beta, \]

\[ \xi = \frac{a_1}{3} - 2 \left( \frac{\beta}{2} \right)^{1/3} \quad \text{for} \quad a_3 = \alpha - \beta. \]

We see that \( a_3 = \alpha + \beta \) gives the greater value for \( \xi \). As the maximum of \( \xi \) exists, at least one of the abscissae \( a_3 = \alpha \pm \beta \) must be in the interval \((0, a_1a_2)\). Since \( D \) is negative for \( a_3 = a_1a_2 \) (where complex roots exist), the abscissa \( a_3 = \alpha + \beta \) surely is in the interval.

Therefore, we may state that

\[ \text{Max } \xi = \frac{a_1}{3} - \frac{1}{(3)^{1/2}} (-p)^{1/2} \quad \text{for} \quad a_3 = \frac{a_1a_2}{3} - \frac{2}{27} a_1^3 + \frac{2}{9} (3)^{1/2} (-p)^{3/2}. \]
Obviously we can combine all cases and write

\[ \text{Max } \xi = \Re \left\{ \frac{a_1}{3} - \frac{1}{(3)^{1/2}} (-p)^{1/2} \right\}. \]

This is obtained for

\[ a_3 = \Re \left\{ \frac{a_1 a_2}{3} - \frac{2}{27} a_1^3 + \frac{2}{9} (3)^{1/2} (-p)^{3/2} \right\}. \]

---

**PERIODIC MOTIONS OF A NON-LINEAR DYNAMIC SYSTEM**

BY H. SERBIN (Purdue University)

1. **Introduction.** It is a matter of common knowledge that many problems of dynamics are non-linear in character and that the linearization of such problems is an expediency adopted in the face of mathematical difficulties attending the non-linearity. Fortunately, the linearized treatment gives useful results in a number of problems. On the other hand, there are cases where the essential characteristics of the phenomenon are altered by the assumption of linearity.

In the absence of general methods for handling non-linear problems, it is desirable to investigate typical non-linear phenomena to as great detail as possible with the intention of arriving at general methods of analysis.

One such problem is van der Pol’s equation

\[ x'' - \epsilon(1 - x^2)x' + x = 0, \quad \left( \dot{x} = \frac{dx}{dt}, \quad x'' = \frac{d^2x}{dt^2} \right). \] (1.1)

This defines the one-dimensional motion of a particle of unit mass suspended on a spring of unit stiffness and subjected to negative damping at low amplitudes and positive damping at high amplitudes.

The existence of periodic solutions of an equation of more general form

\[ x'' + f(x)x' + g(x) = 0 \] (1.2)

was considered by Liénard, Ref. 1, who proved, for \( g(x) = x \) and under fairly general conditions on \( f(x) \) (see Ref. 2), that there exists a periodic solution. The proof used in Ref. 1 was reproduced in Ref. 3 for \( g(x) \neq x \).

In the present paper, the problem is considered from a different point of view. The periodicity of the solution is shown to exist for a wider class of differential equations (1.2) which can best be described and compared as in the following table:

<table>
<thead>
<tr>
<th>( F(x) = \int_0^x f(x) \ dx )</th>
<th>( G(x) = \int_0^x g(x) \ dx )</th>
<th>( f(x), g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lefschetz ( F(\infty) = \infty )</td>
<td>( G(\infty) &gt; 0 )</td>
<td>Symmetric</td>
</tr>
<tr>
<td>Present paper ( F(\infty) &gt; 0 )</td>
<td>( G(\infty) = \infty )</td>
<td>No symmetry required</td>
</tr>
<tr>
<td>( F(\infty) = \infty )</td>
<td>( G(\infty) &gt; 0 )</td>
<td>No symmetry required</td>
</tr>
</tbody>
</table>

*Received July 6, 1949.*