RELATION BETWEEN BERGMAN'S AND CHAPLYGIN'S METHODS OF SOLVING THE HODOGRAPH EQUATION*

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When a perfect gas is in steady irrotational isentropic motion in two dimensions, the stream function ψ satisfies a linear differential equation in which the independent variables are components of velocity. For this 'hodograph equation', general forms of solution have been given by Chaplygin[1] and Bergman[2]. The purpose of this note is to show how Bergman's form of solution can be converted into Chaplygin's. Hereby we obtain the specification of the same solution by means of two quite different series, and are in the position to check the extensive computations which are required (in general) to evaluate either of the series.

The results of §1 are due to Chaplygin[1], Lighthill[3] and Cherry[4]; for proofs of the key-formulae (4), (6), (12) reference may be made to [3] or [4]. For Bergman's form of solution the most convenient reference is v. Mises and Schiffer.[5] The different authors use different notations, and the present paper uses a blend of them.

1. Let the rectangular velocity-components be $\tau^{1/2}\cos\theta$, $\tau^{1/2}\sin\theta$, with the unit of speed so chosen that the limiting speed, at which the pressure vanishes, corresponds to $\tau = 1$. Then the hodograph equation is

$$4(1-\tau)\left(\tau^2\frac{\partial^2\psi}{\partial\tau^2}+\tau\frac{\partial\psi}{\partial\tau}\right)+\frac{4\tau^2}{\gamma-1}\frac{\partial\psi}{\partial\tau}+\left(1-\tau-\frac{2\tau}{\gamma-1}\right)\frac{\partial^2\psi}{\partial\theta^2}=0, \tag{1}$$

where γ is the adiabatic index of the gas. This equation is soluble by separation of the variables, leading to Chaplygin's form of solution

$$\psi = \sum_\nu c_\nu\psi_\nu(\tau)e^{i\nu\theta}, \qquad (c_\nu \text{ constant}) \tag{2}$$

where ν can take any real value except $-2, -3, \cdots$, and

$$\psi_\nu(\tau) = \tau^{\nu/2}F(a_\nu, b_\nu; \nu+1; \tau),$$

$$a_\nu + b_\nu = \nu - \frac{1}{\gamma-1}, \qquad a_\nu b_\nu = -\frac{\nu(\nu+1)}{2(\gamma-1)},$$

F denoting the hypergeometric series.

For τ fixed, $\psi_\nu(\tau)$ is a meromorphic function of ν; its poles are at $\nu = -2, -3, \cdots$, and its residue at $\nu = -m$ is $-h_m\psi_m(\tau)$, where

$$h_m = \frac{\Gamma(a_m)\Gamma(1+m-b_m)}{\Gamma(a_m-m)\Gamma(1-b_m)\Gamma(m)\Gamma(1+m)}. \tag{3}$$

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[1]S. A. Chaplygin, Sci. Ann. Univ. Moscow, Phys-Math. Div. Pub. No. 21 (1904).

[2]S. Bergman, N.A.C.A. Tech. Note No. 972 (1945).

[3]M. J. Lighthill, Proc. Roy. Soc. London, (A) 191, 342 and 352 (1947).

[4]T. M. Cherry, Proc. Roy. Soc. London, (A) 192, 45 (1947).

[5]R. v. Mises and M. Schiffer, *Advances in Applied Mechanics*, vol. I, Academic Press, Inc., New York, 1948, p. 249.

For large values of $|\nu|$, $\psi_\nu(\tau)$ changes character at the point $\tau = \tau_s = (\gamma - 1)/(\gamma + 1)$ at which the coefficient of $\partial^2\psi/\partial\theta^2$ in (1) vanishes; it is (for ν real) monotonic for $0 < \tau < \tau_s$, oscillating for $\tau_s < \tau < 1$. In the first case (with which alone we shall be concerned) there is an asymptotic expansion

$$\psi_\nu(\tau) \sim V(\tau)\delta^\nu e^{\nu\lambda}\left\{1 + \sum_1^\infty p_n(\tau)\nu^{-n}\right\}, \tag{4}$$

valid for all complex ν except the negative integers; here

$$\begin{gathered}\lambda = \tau_s^{-1/2} \operatorname{arc\,tanh}\left(\frac{\tau_s - \tau}{1 - \tau}\right)^{1/2} - \operatorname{arc\,tanh}\left(\frac{1 - \tau/\tau_s}{1 - \tau}\right)^{1/2}, \\ V(\tau) = \frac{(1 - \tau)^{1/4+1/2(\gamma-1)}}{(1 - \tau/\tau_s)^{1/4}}, \qquad \delta = \frac{\alpha^\alpha}{(1 + \alpha)^{1+\alpha}}, \qquad 2\alpha = \tau_s^{-1/2} - 1,\end{gathered} \tag{5}$$

and the $p_n(\tau)$ are determinate functions vanishing at $\tau = 0$. Hence follows the partial fraction expansion

$$\psi_\nu(\tau) = \delta^\nu e^{\nu\lambda}\left\{V(\tau) - \sum_{m=2}^\infty \frac{h_m\delta^m e^{m\lambda}\psi_m(\tau)}{m + \nu}\right\}, \tag{6}$$

valid for $0 \leq \tau < \tau_s$ and all ν. If here we formally expand the last factor in powers of ν^{-1} we must obtain (4), and therefore

$$(-1)^n p_n(\tau)V(\tau) = \sum_2^\infty m^{n-1}h_m\delta^m e^{m\lambda}\psi_m(\tau), \qquad (n = 1, 2, \cdots). \tag{7}$$

Substitute (6) in (2) and interchange the order of the double summation. We obtain, for $0 \leq \tau < \tau_s$,

$$\begin{aligned}\psi = V(\tau) \sum_\nu c_\nu\delta^\nu e^{\nu(\lambda+i\theta)} &- \sum_{m=2}^\infty h_m\delta^m e^{-mi\theta}\psi_m(\tau) \sum_\nu \frac{c_\nu\delta^\nu}{m + \nu} \\ &- \sum_{m=2}^\infty h_m\delta^m e^{-mi\theta}\psi_m(\tau) \sum_\nu \frac{c_\nu\delta^\nu e^{(m+\nu)(\lambda+i\theta)} - c_\nu\delta^\nu}{m + \nu}.\end{aligned}$$

Hence, putting

$$\psi_1(\tau, \theta) = \sum_{m=2}^\infty h_m\delta^m e^{-mi\theta}\psi_m(\tau) \sum_\nu \frac{c_\nu\delta^\nu}{m + \nu}, \tag{8}$$

$$\zeta = \lambda + i\theta, \tag{9}$$

$$\phi_0(\zeta) = \sum_\nu c_\nu\delta^\nu e^{\nu\zeta}, \tag{10}$$

we obtain

$$\begin{aligned}\psi + \psi_1 &= V(\tau) \sum_\nu c_\nu\delta^\nu e^{\nu\zeta} - \sum_{m=2}^\infty h_m\delta^m e^{m\lambda}\psi_m(\tau) \int_0^\zeta \sum_\nu c_\nu\delta^\nu e^{(m+\nu)t-m\zeta}\,dt \\ &= V(\tau)\phi_0(\zeta) - \sum_2^\infty h_m\delta^m e^{m\lambda}\psi_m(\tau) \int_0^\zeta e^{m(t-\zeta)}\phi_0(t)\,dt.\end{aligned} \tag{11}$$

For the justification of these manipulations it is sufficient—apart from the over-riding

condition $0 \leq \tau < \tau_s$—that (i) for all values of ν comprised in (2) and all positive integers m, $|\nu + m|$ has a positive lower bound, and (ii) the series $\sum |c_\nu \delta^\nu e^{\nu\zeta}|$ converges in some strip $\zeta_1 \leq \operatorname{Re} \zeta \leq 0$; the proof rests essentially upon the estimations

$$\psi_\nu(\tau) = V(\tau)\delta^\nu e^{\nu\lambda}\{1 + O(\nu^{-1})\}, \qquad 2\pi h_m = \delta^{-2m}\{1 + O(m^{-1})\}, \tag{12}$$

of which the former is the first approximation derived from (4).

We note that $\psi_1(\tau, \theta)$, as defined in (8), is a solution of (1) in Chaplygin's form.

2. To convert (11) into Bergman's form of solution we expand the factor $e^{m(t-\zeta)}$ and rearrange the resulting double sum. After an appeal to (7) this gives

$$\psi + \psi_1 = V(\tau)\phi_0(\zeta) - \sum_2^\infty h_m \delta^m e^{m\lambda}\psi_m(\tau) \int_0^\zeta \phi_0(t)\, dt \sum_1^\infty m^{n-1}(t - \zeta)^{n-1}/(n-1)!$$

$$= V(\tau)\phi_0(\zeta) - \int_0^\zeta \sum_{n=1}^\infty (t - \zeta)^{n-1}\phi_0(t)\, dt/(n-1)! \cdot \sum_{m=2}^\infty m^{n-1} h_m \delta^m e^{m\lambda}\psi_m(\tau)$$

$$= V(\tau)\phi_0(\zeta) + \int_0^\zeta \sum_{n=1}^\infty \frac{(\zeta - t)^{n-1} p_n(\tau) V(\tau)}{(n-1)!}\phi_0(t)\, dt. \tag{13}$$

The transformation is valid provided the series

$$\sum_2^\infty h_m \delta^m e^{m\lambda}\psi_m(\tau) e^{m|\zeta|}$$

converges absolutely, and by (12) this is secured if $|\zeta| + 2\lambda$ is negative; hence from (9), it is sufficient that λ be negative (as it is for $0 < \tau < \tau_s$) and that

$$-3^{1/2}|\lambda| < \theta < 3^{1/2}|\lambda|. \tag{14}$$

On the left of (13) $\psi + \psi_1$ is a solution of Chaplygin's form, and on the right we have this expressed in Bergman's form* in terms of an arbitrary analytic function $\phi_0(\zeta)$. The identification not merely of form but of content will be complete provided Bergman's G_n and the present p_n are related by

$$G_n = (-2)^n p_n. \tag{15}$$

Now if, as in [5], we examine the conditions that the form on the right of (13) be a solution, with $\phi_0(\zeta)$ remaining arbitrary, we find that the derivative of p_n *must* be determined entirely by p_{n-1}, so that p_n is determined apart from an additive constant. This constant is, in the preceding work, determined by the condition $p_n = 0$ for $\tau = 0$, while in [5] the condition is taken to be $G_n = 0$ for $\lambda = -\infty$; and these conditions agree since to $\tau = 0$ corresponds by (5) $\lambda = -\infty$. Hence (15) expresses merely the same function in two different notations.

In conclusion, it may be remarked that the conditions assumed in proving (13) are, in one respect, more restrictive than those which validate Bergman's form of solution on the right; for our conditions imply that $\phi_0(\zeta)$ is regular in a strip $\zeta_1 < \operatorname{Re} \zeta < 0$, whereas Bergman requires only regularity in a partial neighbourhood of $\zeta = 0$. Against this must be set the fact that Bergman's form is established only when θ is restricted as in (14), while in the Chaplygin form θ is unrestricted.

*See particularly [5], p. 258, un-numbered equation following (4); here ψ^* is defined in (1.6), where $z^{-1/2}$ is the same as $V(\tau)$ of the present paper.