DIFFICULTIES WITH PRESENT SOLUTIONS OF THE HALLÉN INTEGRAL EQUATION*

BY

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I. Introduction. In a recent paper,[1] rather serious discrepancies were shown to exist between values of broadside absorption gain and back-scattering cross section as found from experiment, and those predicted from Hallén's first-order solution[2] as modified by King and Middleton.[3] In this paper these discrepancies and certain additional shortcomings with the present solutions to Hallén's integral equation will be discussed.

II. The first-order current distributions. It can be shown that the current distribution on a receiving dipole antenna is given by[4]

$$I_r(z) = I_E(z) - I_v(z), \tag{1}$$

where $I_E(z)$ is the current distribution due to the external field, E_i, on the antenna with zero load (shorted), and $I_v(z)$ is the current distribution along the antenna when driven by a voltage, V_L, equal to the voltage drop across the receiving antenna load. The distribution of Eq. (1) need be considered only in the cases where scattering behavior is desired. All the other properties of a receiving antenna usually of interest, such as absorption gain, impedance, and effective length, are determined by the driven current distribution alone.

The two different current distributions involved are given as follows:

a. The transmitting dipole. The first-order solution of Hallén's integral equation for the current distribution on a center-fed dipole of length $2h$ and radius a is given by

$$I_v(z) = j2\pi V_0 f_v(z)(\zeta\psi_t H_2)^{-1}, \tag{2}$$

where V_0 is the voltage at the terminals, $\zeta = 120\pi$ ohms, and

$$f_v(z) = f_v'(z) + jf_v''(z) = b_1 \cos \beta z - b_2 \sin \beta \mid z \mid - C(z) \sin \beta h + S(z) \cos \beta h, \tag{3a}$$

$$b_1 = [2\psi_t + E(h)] \sin \beta h - S(h), \tag{3b}$$

$$b_2 = [2\psi_t + E(h)] \cos \beta h - C(h), \tag{3c}$$

$$H_2 = H_2' + jH_2'' = [\psi_t + E(h)] \cos \beta h - C(h). \tag{4}$$

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[1]S. H. Dike and D. D. King, *The cylindrical dipole receiving antenna*, Tech. Report No. 12, Radiation Laboratory, Johns Hopkins University, 1951. (Submitted to *Proc. of I.R.E.* for publication).

[2]E. Hallén, *Theoretical investigations into the transmitting and receiving properties of antennas*, Nova Acta, Royal Soc. Sciences (Uppsala) **11**, 1-44 (1938).

[3]R. W. P. King and D. Middleton, *The cylindrical antenna: current and impedance*, Q. Appl. Math. **3**, 302-335 (1946).

[4]R. W. P. King, H. Mimno, and A. Wing, *Transmission lines, antennas and wave guides*, McGraw-Hill Book Co., New York, p. 163; 1945.

The functions $C(z)$, $S(z)$ and $E(z)$ are defined in Ref. 3. The function ψ_t is the expansion parameter. It is determined in the King-Middleton method by considering a function defined by

$$\Psi_t(z) = \int_{-h}^{h} g_t(z, s) r^{-1} e^{-j\beta r}\, ds, \tag{5}$$

where

$$r = [(z - s)^2 + a^2]^{1/2}, \tag{6}$$

and

$$g_t(z, s) = f(s)/f(z), \tag{7}$$

such that

$$I_v(z) = I_{0v} f(z);\ I_v(s) = I_{0v} f(s), \tag{8}$$

where I_{0v} is the terminal current in the driven dipole. In the limit of vanishing dipole radius the driven dipole current distribution can be shown to be[5]

$$I_v(z) = I_{0v} (\sin \beta h)^{-1} \sin \beta(h - |z|). \tag{9}$$

King and Middleton choose the function $f(z) = \sin \beta(h - |z|)$ giving

$$g_t(z, s) = \sin \beta(h - |s|)/\sin \beta(h - |z|), \tag{10}$$

and

$$\Psi_t(z) = \frac{C(z) \sin \beta h - S(z) \cos \beta h}{\sin \beta(h - |z|)}. \tag{11}$$

King and Middleton then argue that $\Psi_t(z)$ is predominately real, and that a suitable expansion parameter may be found by setting

$$\Psi_t(z) = |\Psi_t(z_0)| = \psi_t, \tag{12}$$

where z_0 is chosen so that $\Psi_t(z_0)$ is a good approximation to $\Psi_t(z)$ over most of the antenna. Accordingly, they choose

$$\psi_t = \begin{cases} |C(0) \sin \beta h - S(0) \cos \beta h| (\sin \beta h)^{-1}, & \beta h \leq \frac{\pi}{2}, \\ \left| C\left(h - \frac{\lambda}{4}\right) \sin \beta h - S\left(h - \frac{\lambda}{4}\right) \cos \beta h \right|, & \beta h \geq \frac{\pi}{2}. \end{cases} \tag{13}$$

b. The unloaded receiving dipole. The first-order current distribution on a shorted dipole antenna placed parallel to the electric vector of a plane-wave, far-zone field is

$$I_E(z) = j4\pi E_i f_E(z) (\beta \zeta \psi_r H_1)^{-1}, \tag{14}$$

where

$$f_E(z) = 2\psi_r \cos \beta z + E(z) \cos \beta h - C(z) + C(h) - \cos \beta h[2\psi_r + E(h)], \tag{15}$$

$$H_1 = H_1' + jH_1'' = \psi_r \cos \beta h + E(h) \cos \beta h - C(h). \tag{16}$$

[5]S. Schelkunoff, *Electromagnetic waves*, D. Van Nostrand Co., New York, 1943, p. 142.

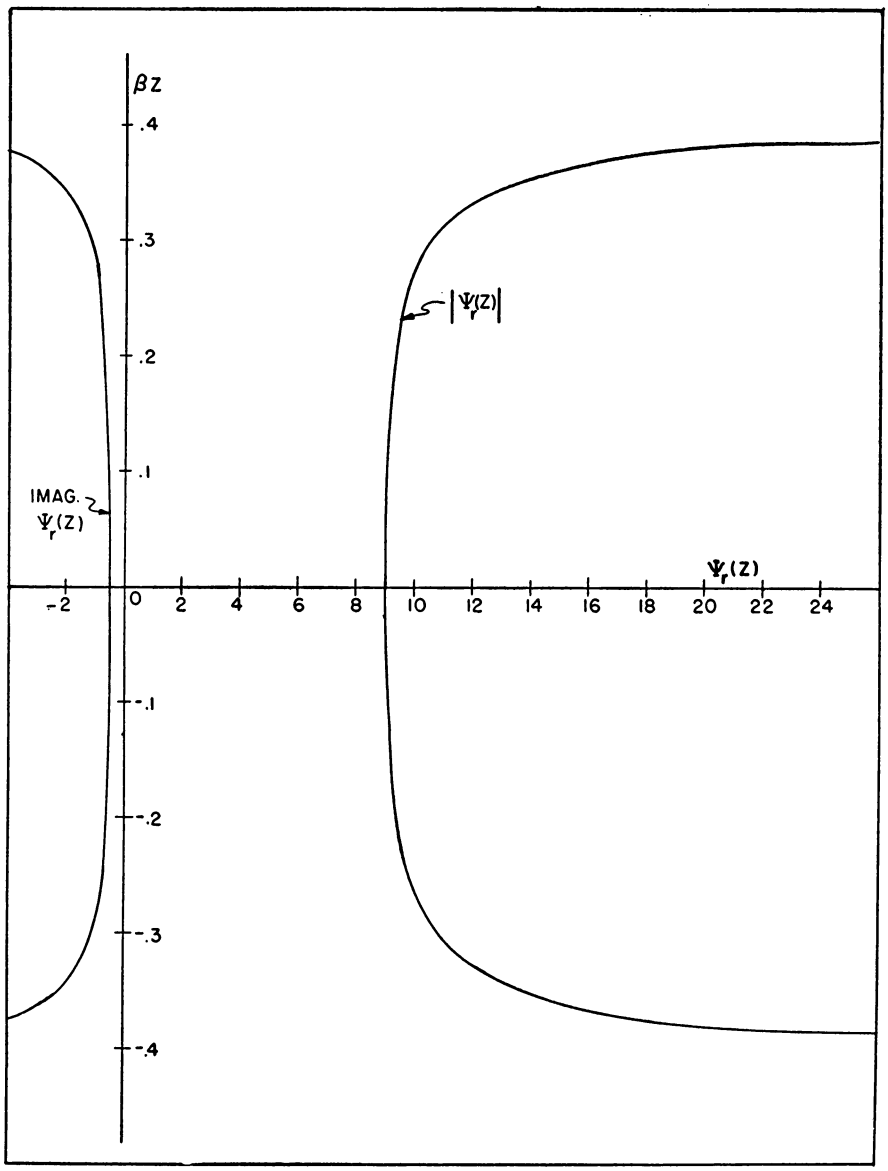

FIG. 1. The Function $\Psi_r(z)$ for $\beta h = 0.4$.
$\Omega = 10$

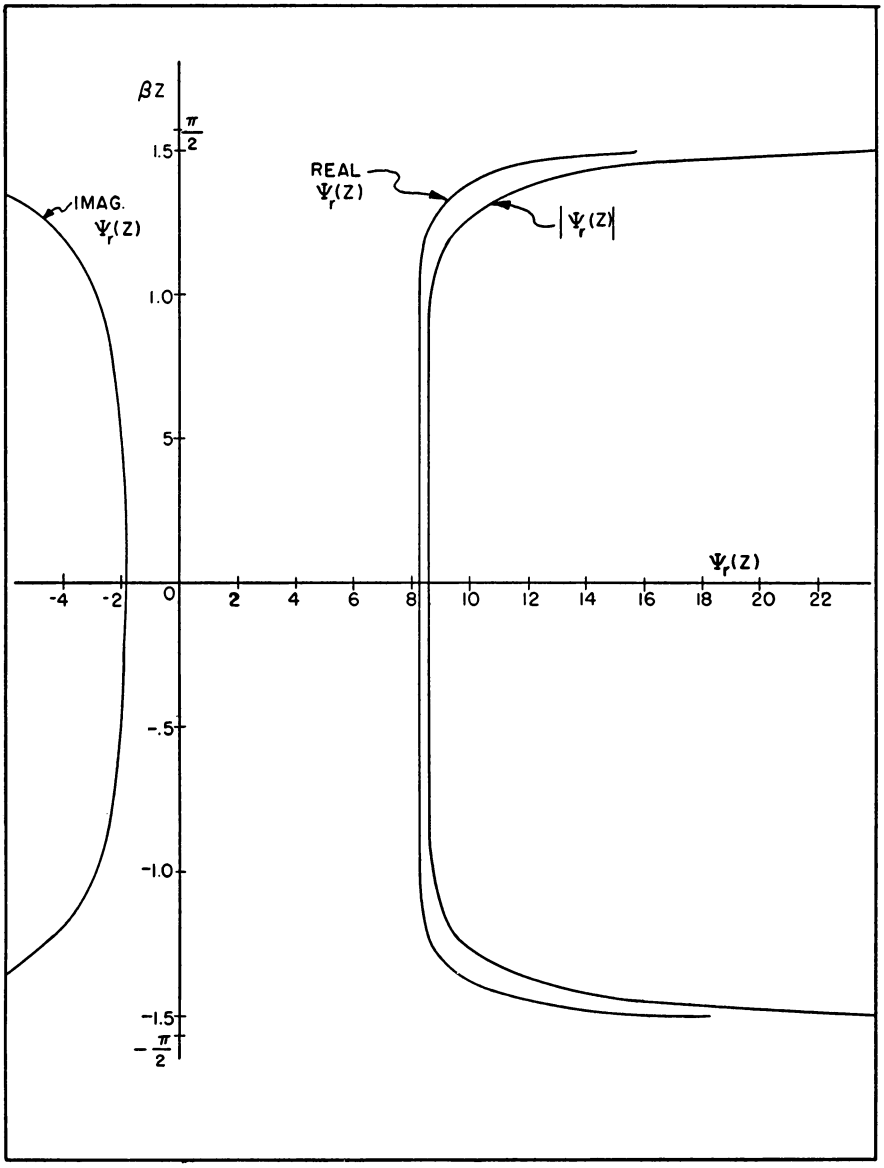

FIG. 2. The Function $\Psi_r(z)$ for $\beta h = \pi/2$
$\Omega = 10$

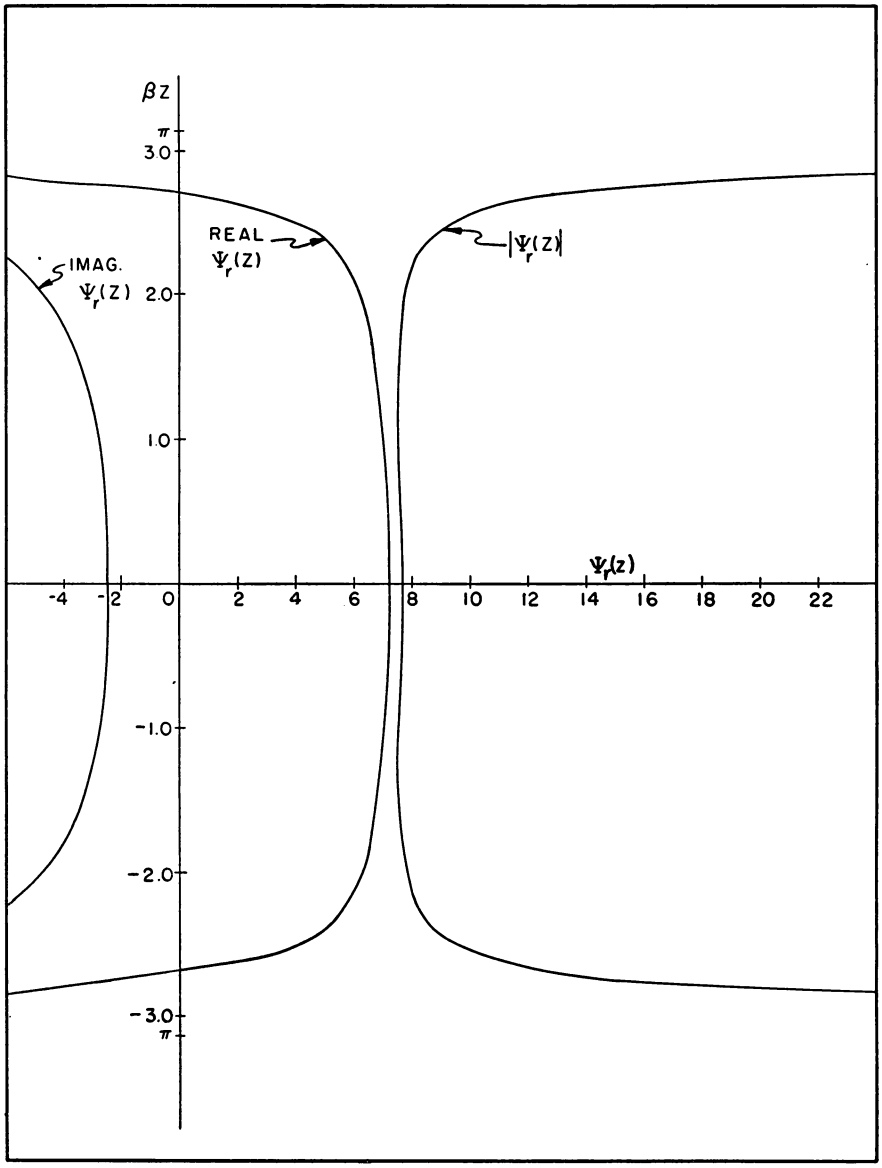

FIG. 3. The Function $\Psi_r(z)$ for $\beta h = \pi$
$\Omega = 10$

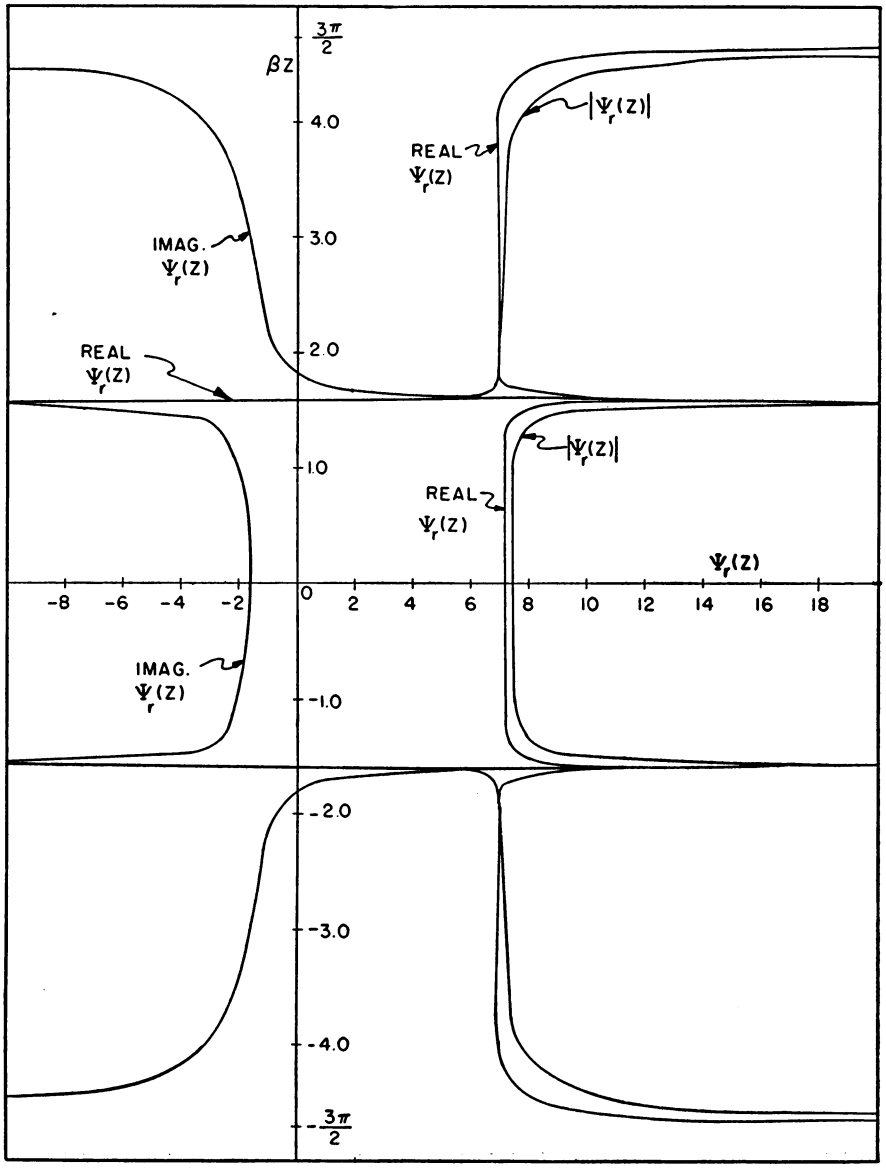

FIG. 4. The Function $\Psi_r(z)$ for $\beta h = \frac{3\pi}{2}$

$\Omega = 10$

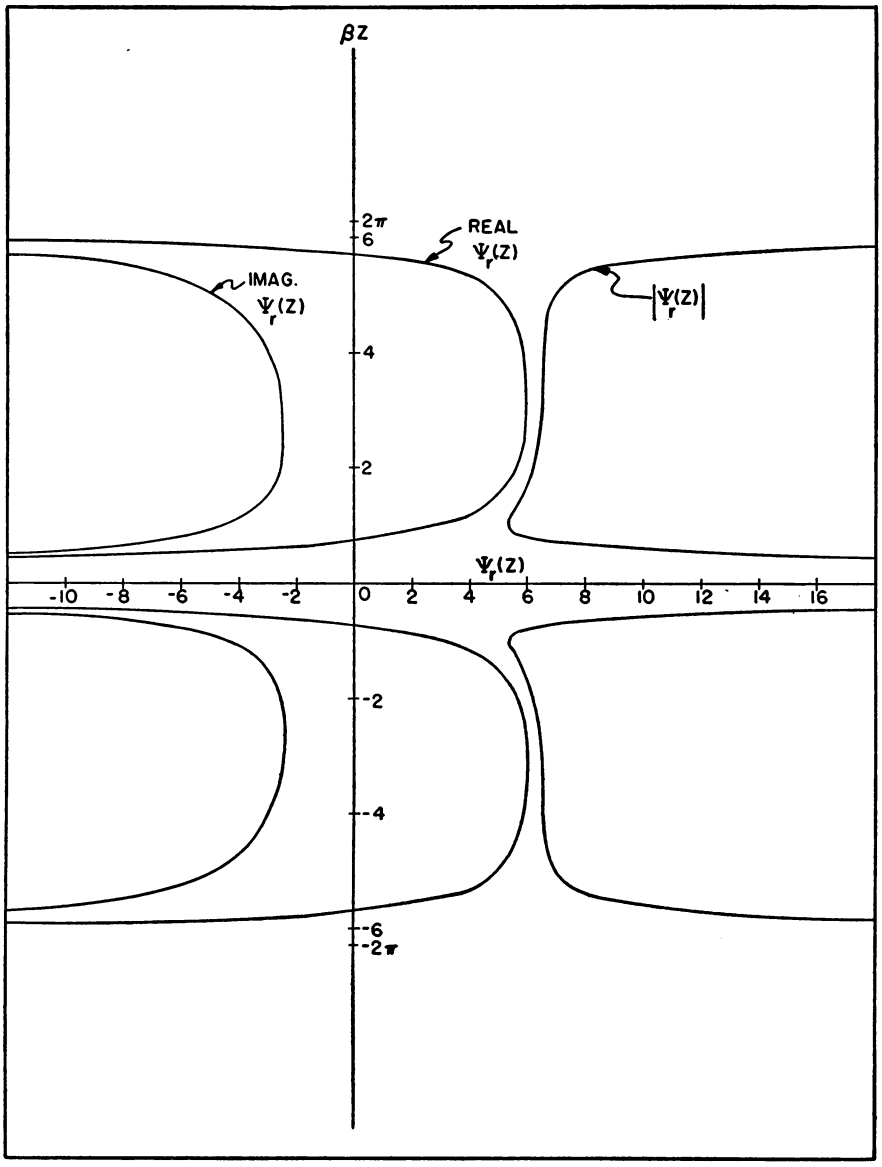

FIG. 5. The Function $\Psi_r(z)$ for $\beta h = 2\pi$
$\Omega = 10$

For the indefinitely thin shorted dipole, the current distribution is given by[6]

$$I_E(z) = I_{0E}(1 - \cos \beta h)^{-1}(\cos \beta z - \cos \beta h). \tag{17}$$

By the arguments of King and Middleton one is led to choose $f(z) = \cos \beta z - \cos \beta h$ so that

$$g_r(z, s) = \frac{\cos \beta s - \cos \beta h}{\cos \beta z - \cos \beta h}, \tag{18}$$

and $\Psi_r(z)$ becomes

$$\Psi_r(z) = \frac{C(z) - E(z) \cos \beta h}{\cos \beta z - \cos \beta h}. \tag{19}$$

This function is plotted in Figs. 1 to 5 for various values of βh and for $\Omega = 2 \log 2h/a = 10$. By consideration of these figues and using the arguments of King and Middleton, one finds that a suitable choice of ψ_r is

$$\psi_r = \begin{cases} | C(0) - E(0) \cos \beta h | (1 - \cos \beta h)^{-1}, & \beta h \leq \pi, \\ \frac{1}{2} | C(0) + E(0) |, & \beta h \geq \pi. \end{cases} \tag{20}$$

In the limit of very small βh, $\psi_t = \Omega - 2$ and $\psi_r = \Omega - 1$.

For the broadside case both the currents $I_v(z)$ and $I_E(z)$ are even functions. Equation (1) may now be expressed in terms of the foregoing as

$$I_r(z) = \frac{j4\pi E_i}{\beta \zeta \psi_r H_1} \left\{ f_E(z) - \frac{j2\pi Z_e f_E(0) f_v(z)}{\zeta \psi_t H_2} \right\}, \tag{21}$$

where

$$Z_e = Z_a Z_L/(Z_a + Z_L). \tag{22}$$

Z_a is the antenna impedance defined by

$$Z_a = V_0/I_{0v}. \tag{23}$$

Z_L is the receiving antenna load impedance. For matched load

$$Z_e = | Z_a |^2/2R_a, \qquad Z_L = Z_a^*. \tag{24}$$

III. Formulas for broadside gain, effective length and back-scattering cross section. Three equivalent expressions for absorption gain were derived in Ref. 1. They are

$$G_t = \frac{\pi | Z_a |^2 | \beta g_v(h) |^2}{\zeta \psi_t^2 R_a | H_2 |^2}, \tag{25}$$

$$G_t = \frac{| \beta g_v(h) |^2}{2\psi_t T}, \tag{26}$$

$$G_t = \frac{\pi \zeta | d |^2}{R_a \lambda^2}, \tag{27}$$

[6]King, Mimno, and Wing, *loc. cit.*

where

$$T = H_2''f_v'(0) - H_2'f_v''(0), \tag{28}$$

and where, for the broadside case,*

$$\begin{aligned}\beta g_v(h) = \beta \int_{-h}^{h} f_v(z)\, dz &= (4\psi_t - 2\Omega)(1 - \cos \beta h) \\ &+ 2 \operatorname{Ein} 2\beta h + \cos \beta h(\operatorname{Ein} 2\beta h - \operatorname{Ein} 4\beta h - 4 \operatorname{Ein} \beta h) - j \sin \beta h \operatorname{Ein} 4\beta h.\end{aligned} \tag{29}$$

R_a is the radiation resistance defined as the real part of Eq. (23). The symbol d denotes the effective length of the antenna. For the broadside case[7]

$$d = \frac{1}{I_{0v}} \int_{-h}^{h} I_v(z)\, dz, \tag{30}$$

or

$$| d |/\lambda = | \beta g_v(h) | (2\pi | f_v(0) |)^{-1}, \tag{31}$$

or

$$| d |/\lambda = | Z_a | | \beta g_v(h) | (\zeta \psi_t | H_2 |)^{-1}. \tag{32}$$

The relation for the back-scattering cross section, σ, derived in Ref. 1 is

$$\sigma/\lambda^2 = | \beta g_E(h) - BZ_e f_E(0)\beta g_v(h) |^2 (\pi \psi_r^2 | H_1 |^2)^{-1}, \tag{33}$$

where

$$B = j2\pi(\zeta \psi_t H_2)^{-1} = 2\pi(H_2'' + jH_2')(\zeta \psi_t | H_2 |^2)^{-1}, \tag{34}$$

and

$$\begin{aligned}\beta g_E(h) = \beta \int_{-h}^{h} f_E(z)\, dz = \sin \beta h(4\psi_r - 2\Omega - 4 \log 2 \\ - j \operatorname{Ein} 4\beta h + 2 \operatorname{Ein} 2\beta h + \operatorname{Ein} 4\beta h) \\ + \cos \beta h[2\beta h(\Omega - 2\psi_r + 2 \log 2) - \beta h \operatorname{Ein} 4\beta h - 2\beta h \operatorname{Ein} 2\beta h \\ - j \operatorname{Ein} 4\beta h - 2 \sin 2\beta h - 2j(\cos 2\beta h - 1)].\end{aligned} \tag{35}$$

IV. The required equality of ψ_r and ψ_t. Although the method of King and Middleton yields a different expansion parameter in the receiving case from that found by them for the driven dipole, it is necessary, in order to have a consistent theory for the receiving dipole, that both expansion parameters be identical. This can be seen from a comparison of the two equivalent definitions for effective length. In addition to Eq. (30), effective length may be defined by

$$d = I_{0E} Z_a / E_i\,. \tag{36}$$

*The function $\operatorname{Ein}(x)$ is defined in the Appendix.

[7]S. H. Dike, *The effective length of antennas*, Tech. Report No. 13, Radiation Laboratory, Johns Hopkins University, 1951. (Submitted to I.R.E.).

From (30) and (36) one obtains the interesting equality that

$$\frac{I_{0E}}{E_i} = \frac{1}{V_0} \int_{-h}^{h} I_v(z)\, dz. \tag{37}$$

Substituting in (37) from (2), (14), and (29), one obtains

$$\frac{2f_E(0)}{\psi_r H_1} = \frac{\beta g_v(h)}{\psi_t H_2}. \tag{38}$$

This relation is true if $\psi_t = \psi_r$, because when this is so, $H_1 = H_2$ and $2f_E(0)$ does indeed equal $\beta g_v(h)$ for all orders of solution. A second relation is possible between ψ_r and ψ_t which satisfies (38), but this relation is dependent upon the order of the solution. This means that the expansion parameter would be a function of the number of terms retained in the solutions for the currents. This is obviously undesirable. In any case ψ_r must equal ψ_t for the zero order solution.

This fact constitutes the first difficulty encountered in the King-Middleton method. It is difficult to say which of the two parameters, ψ_t or ψ_r, is the better. It appears that neither is particularly good.[1]

V. The behavior of the theory for short dipoles. It was pointed out in Ref. 1 that the values of gain obtained from Eq. (26) do not reduce in the limit of decreasing βh to the value 1.5. One would expect the value 1.5 as being the correct one for finite Ω because for very small βh the current distribution must be essentially linear. This does not imply that either the radiation resistance or the effective length should reduce to those of the indefinitely thin short dipole. For very small βh, Eq. (26) becomes

$$G_t = \frac{3}{2}\left[\frac{(2\psi_t - \Omega + 3)^2}{3\psi_t^2 - 2\psi_t(\Omega - 2 - 2\log 2)}\right], \qquad \beta h \ll 1. \tag{39}$$

This result is independent of βh but remains a function of the expansion parameter. Since for small βh, $\psi_t = \Omega - 2$, Eq. (39) becomes

$$G_t = \frac{3}{2}\left[\frac{(\Omega - 1)^2}{(\Omega - 2)(\Omega - 2 + 4\log 2)}\right], \qquad \beta h \ll 1. \tag{40}$$

This is not 1.5 except for $\Omega \to \infty$. It is not evident that the difficulty would be removed by retaining more terms of the series solution. Various values of short dipole gain can be obtained from (39) by using the expansion parameters of previous authors. This is shown for $\Omega = 10$ in Table I, where the value of the expansion parameter for small βh is given.

TABLE I

Author	ψ	G_t	*Percent Error*
Hallén	Ω	1.5114	0.76
Gray[8]	$\Omega - 2 + \log 4$	1.4835	1.10
King-Middleton	$\Omega - 2$	1.4098	6.01

If the parameter ψ_r is used, where $\psi_r = \Omega - 1$ for $\beta h \ll 1$, the value of G_t at $\Omega = 10$ is 1.464 which is an improvement over the use of ψ_t.

[8]M. C. Gray, *A modification of Hallén's solution of the antenna problem*, J. Appl. Phys. 15, 61-65 (1944).

The fact that the bracket of (39) should be unity in the first-order theory gives

$$\psi^2 - \psi(2\Omega - 8 + 4 \log 2) + \Omega^2 - 6\Omega + 9 = 0,$$

or

$$\psi = \Omega - 2.6138 \pm 0.879\,(\Omega - 2.807)^{1/2}. \tag{41}$$

Of the two values allowed, the larger is probably the one that should be chosen.

A similar situation results in the value of σ/λ^2 for very short matched-loaded dipoles. The theory should reduce to the value $9/16\pi$. In the limit of decreasing βh, Eq. (33) for the matched-loaded case becomes

$$\frac{\sigma}{\lambda^2} = \frac{9}{16\pi}\left[\frac{\psi_t(2\psi_t - \Omega + 3)(2\psi_r - \Omega + 3)}{\psi_r^2(3\psi_t - 2\Omega + 4 + 4 \log 2)}\right]^2. \tag{42}$$

If the requirement is made that $\psi_t = \psi_r$ then

$$\frac{\sigma}{\lambda^2} = \frac{9}{16\pi}\left[\frac{(2\psi - \Omega + 3)^2}{3\psi^2 - 2\psi(\Omega - 2 - 2 \log 2)}\right]. \tag{43}$$

The bracket of (43) is identical to that of (39) leading to the same requirement on ψ given by (41). It appears then that an additional requirement should be imposed on the expansion parameter that has not previously been considered.

It is also of interest to consider the value of impedance in the limit of very small βh. The impedance is given by

$$Z_a = V_0/I_{0v} = -j\zeta\psi_t H_2[2\pi f_v(0)]^{-1}, \tag{44}$$

which for $\beta h \ll 1$, becomes

$$Z_a = R_a + jX_a = \frac{-60\psi_t[3\psi_t + 2j(\beta h)^3]}{3\beta h(2\psi_t + x) + j(\beta h)^4}, \tag{45}$$

where

$$x = 2 + 2 \log 2 - \Omega. \tag{46}$$

Separating real and imaginary parts of (45):

$$R_a = 20\psi_t(\beta h)^2(3\psi_t + 2x)(2\psi_t + x)^{-2}, \tag{47}$$

and

$$X_a = -\frac{60\psi_t^2}{\beta h(2\psi_t + x)}. \tag{48}$$

Since $\psi_t = \Omega - 2$ for $\beta h \ll 1$,

$$R_a = \frac{20(\beta h)^2}{1 + (2 \log 2)^2/y}, \tag{49}$$

where

$$y = \Omega^2 + (\log 2 - 1)(\log 2 - 1 + 4\Omega), \tag{50}$$

and

$$X_a = \frac{-60}{\beta h}\left[\frac{(\Omega - 2)^2}{\Omega - 2 + 2 \log 2}\right]. \tag{51}$$

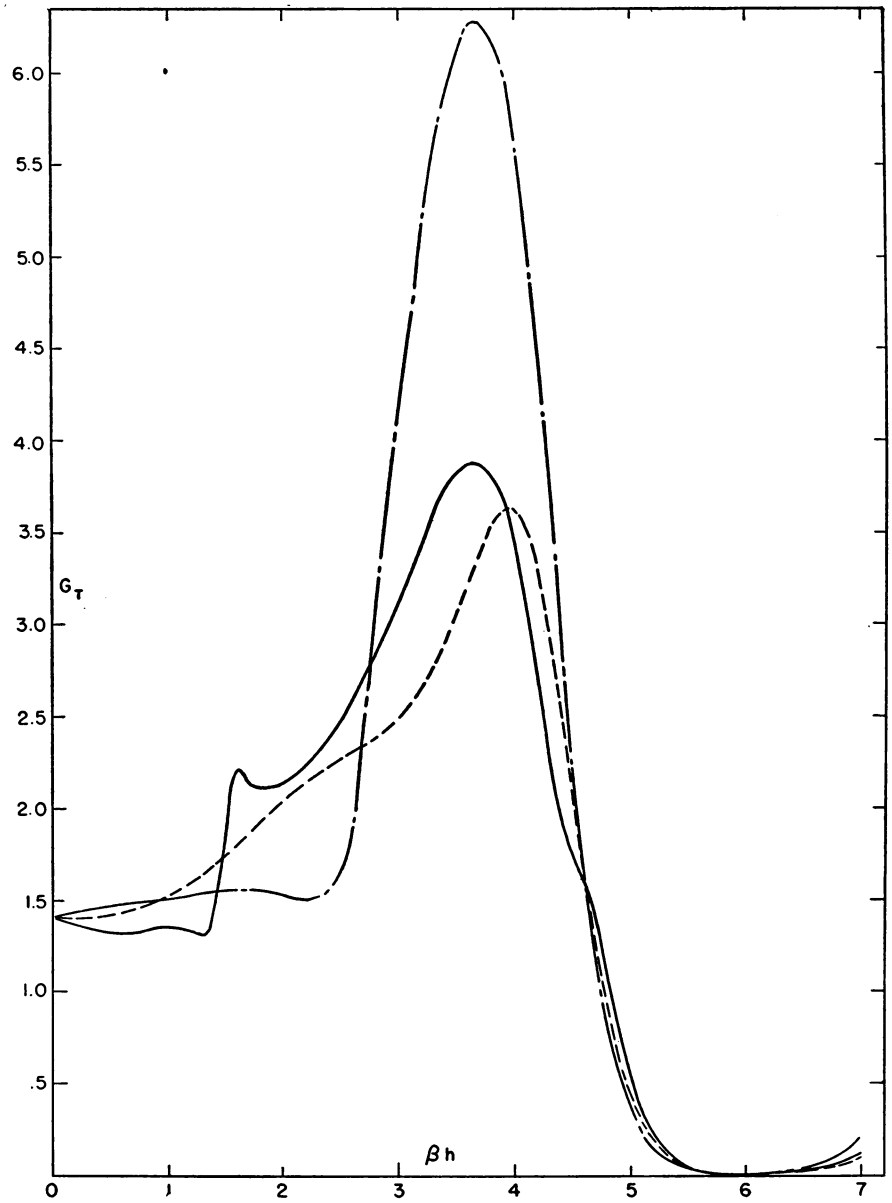

FIG. 6. Absorption Gain for $\Omega = 10$

——— Equation (25) using second-order $|Z_a|^2/R_a$

-------- First-order theory

— · — Equation (27) using second-order R_a

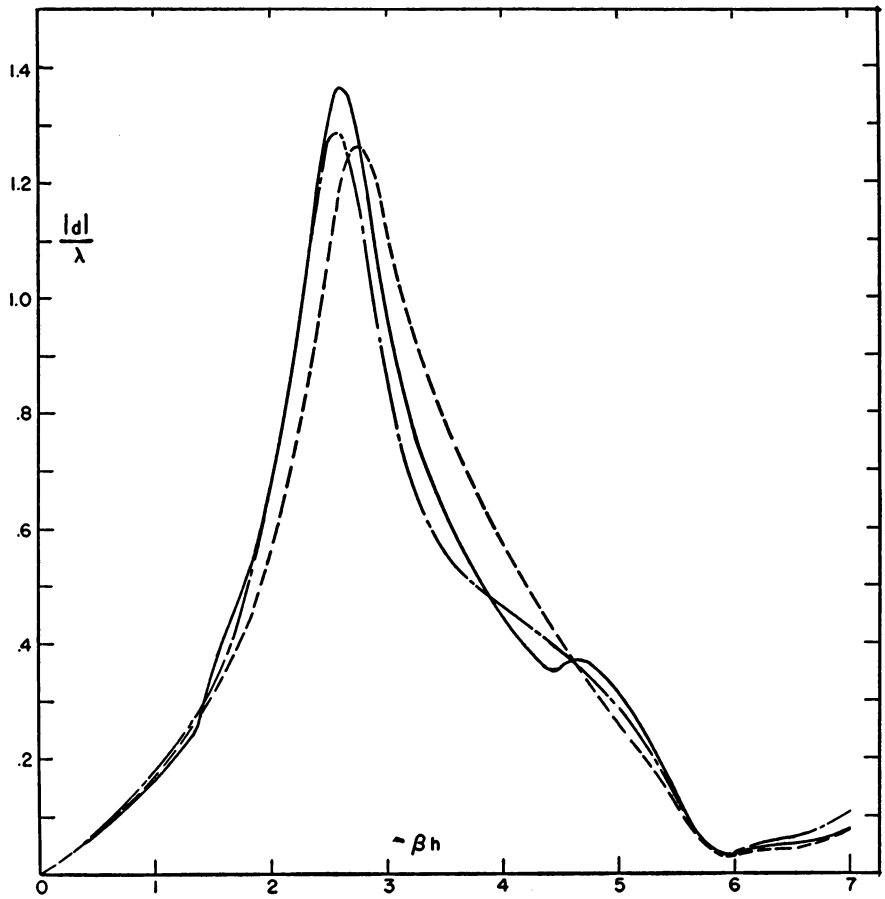

FIG. 7. Effective length for $\Omega = 10$

-------- First-order theory

— · — Equation (27) using second-order R_a

——— Equation (32) using second-order Z_a

For a value of ψ which satisfies (41), R_a becomes

$$R_a = 20(\beta h)^2(2\psi - \Omega + 3)^2(2\psi - \Omega + 2 + 2 \log 2)^{-2}. \tag{52}$$

Using ψ from (41) for $\Omega = 10$, $R_a = 20(\beta h)^2(0.941)$. The corresponding value from the King-Middleton relation (49) is $R_a = 20(\beta h)^2(0.978)$. There is a difference of about four percent between these two values.

The corresponding values of X_a for $\Omega = 10$ are $X_a = -(7.375)60/\beta h$ for ψ satisfying (41), and $X_a = -(6.819)60/\beta h$ for the King-Middleton expression (51). The difference here is about eight percent.

It is of interest to compare these results with the reactance computed from the static capacitance between two cylinders placed end to end in air, and separated by a distance

which is negligible compared with their individual lengths. This reactance is given by[9]

$$X_{as} = \frac{-\zeta(\Omega - \log 12)}{2\pi\beta h} = \frac{-60(\Omega - 2.485)}{\beta h}. \tag{53}$$

For $\Omega = 10$, Eq. (53) gives $X_{as} = -(7.515)60/\beta h$. This result may be compared with those obtained from Eq. (48) listed in Table II.

TABLE II

ψ	$\frac{-\beta h X_a}{60}$ for $\Omega = 10$, $\beta h \ll 1$
Ω (Hallén)	7.470
from Eq. (41)	7.375
$\Omega - 2 + \log 4$ (Gray)	7.246
ψ_r	7.114
$\Omega - 2$ (King-Middleton)	6.819

VI. The use of the King-Middleton values of impedance. It was thought that improved values of the various antenna properties might be obtained by using the second-order impedance values of King and Middleton in the expressions for gain, back-scattering cross section, and effective length. Figure 6 shows the absorption gain, G_t, computed from the straight first-order theory, and also as computed from Eq. (25) using second-order values of $|Z_a|^2/R_a$, and from Eq. (27) using first-order effective length and second-order R_a. As can be seen from this figure, very different results are obtained when the King-Middleton values are used. The use of Eq. (27) with second-order R_a would seem to indicate that the King-Middleton values for the radiation resistance are too large in the region of $\beta h = 2$, and are too small in the region of $\beta h = 3.5$. The use of Eq. (25) with second-order $|Z_a|^2/R_a$ yields results which are certainly contrary to fact near $\beta h = 1.5$ and $\beta h = 4.5$. The "bulge" in the first-order theory near $\beta h = 2$ is contrary to experimental data and disappears if ψ_r is used instead of ψ_t.[1]

Figure 7 is a comparison of the first-order theory for effective length given by Eq. (31), the relation of Eq. (32) using second-order Z_a, and Eq. (27) using second-order R_a and first-order G_t. The use of (32) with second-order Z_a gives results which are unreasonable near $\beta h = 1.5$ and $\beta h = 4.5$.

Figure 8 shows the back-scattering cross section for matched load according to the first-order theory and also from Eq. (33) where second-order King-Middleton values are used for Z_e. The latter curve behaves strangely near $\beta h = 1.5$. Neither curve represents experiment, particularly in the region above $\beta h = 4$.[1]

These three figures show that it is not permissible to use second-order King-Middleton values of impedance in first-order formulas. This may be due to the unknown behavior of the series solution as regards convergence,[10] or it may be that the second-order impedance values of King and Middleton are not good. This latter case would imply that the expansion parameter ψ_t can be better chosen.

VII. The problem of choosing the expansion parameter. The integral equation of

[9] R. W. P. King and C. Harrison, Jr., *The impedance of short, long and capacity loaded antennas with a critical discussion of the antenna problem*, J. Appl. Phys., **17**, 170 (1944).

[10] S. Schelkunoff, *Concerning Hallén's integral equation for cylindrical antennas*, Proc. I.R.E., **33**, 872 (1945).

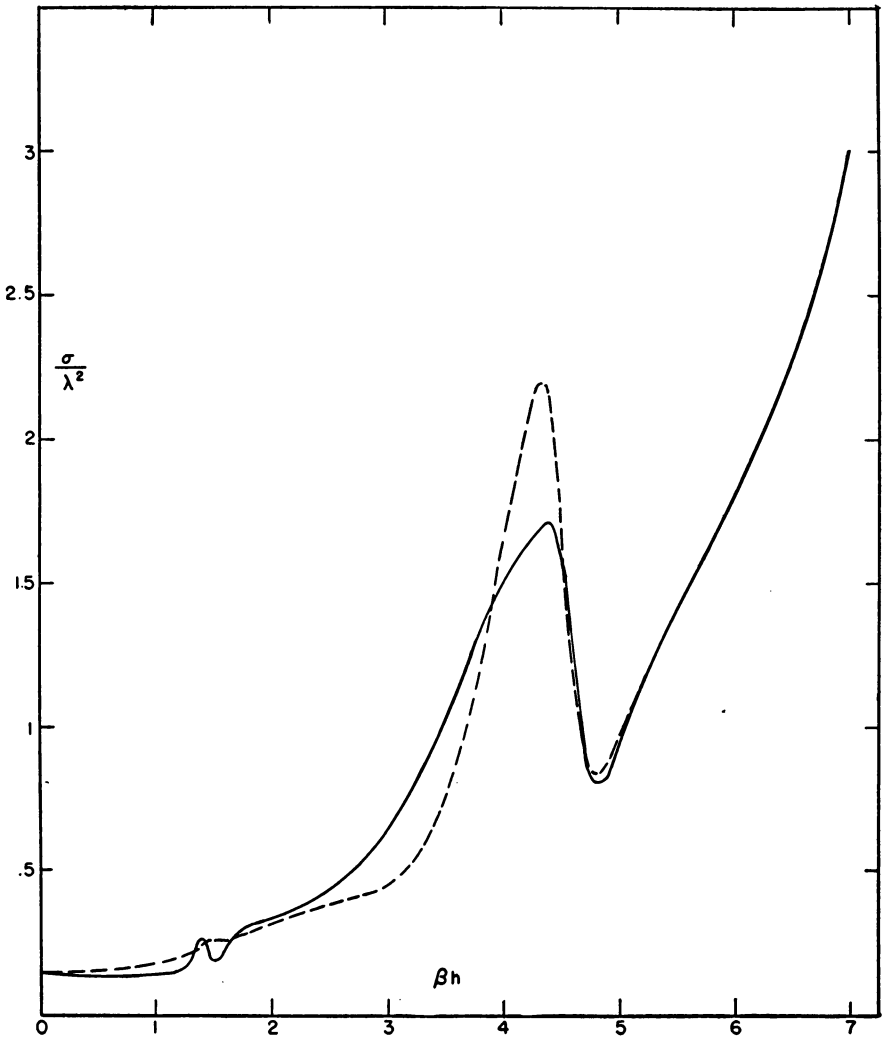

FIG. 8. Back-scattering cross-section for $\Omega = 10$, $Z_L = Z_a^*$
——— First-order theory
——— Equation (33) using second-order $|z_a|^2 R_a$

Hallén is known to be a sufficiently accurate formulation of the problem. It has been examined by many workers,[11,12,13] and has been shown to contain approximations only to the order $(a/h)^2$.[10] Hallén proposed an iterative process for solving this equation and obtained a series solution.[2] Modified solutions have been proposed by Miss Gray[8] and by King and Middleton.[3] These have consisted essentially of modifying the expansion

[11]D. Middleton and R. W. P. King, *The thin cylindrical antenna: a comparison of theories*, J. Appl. Phys. **17**, 273-284 (1946).

[12]L. Brillouin, *The antenna problem*, Q. Appl. Math. **1**, 201-214 (1943).

[13]S. Schelkunoff, *Antenna theory and experiment*, J. Appl. Phys. **15**, 54-60 (1944).

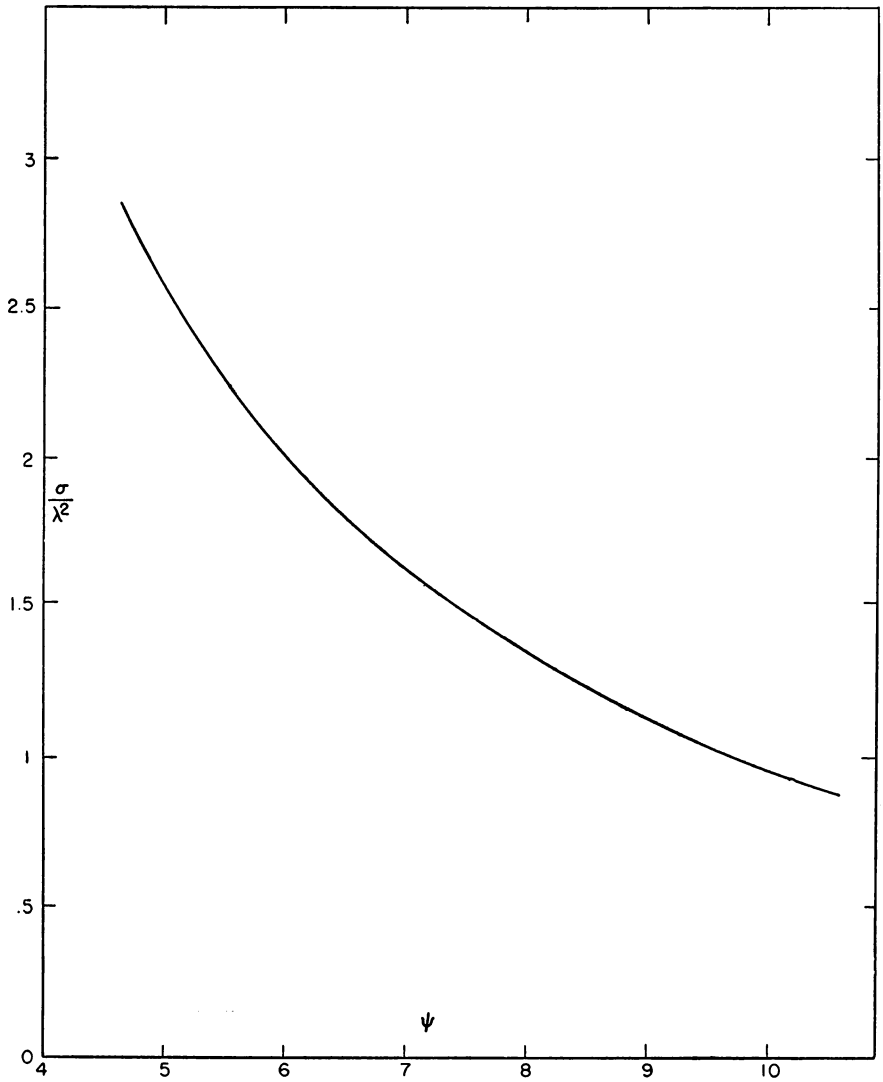

FIG. 9. Back-scattering cross-section at $\beta h = 2\pi$ for $\Omega = 10$, $Z_L = 0$, as a function of expansion parameter

parameter used. Hallén[14] claims that such modifications have no mathematical foundation, and this may be so. However, the method of King and Middleton does appear reasonable from the standpoint of physical reasoning. The choice of a trial function which is known to be representative of the current distribution in the limit of vanishing dipole radius seems to have some merit, although the arguments as to why this should yield a better solution have been attacked by Hallén.[15] Nevertheless, if it is thought

[14]E. Hallén, *Admittance diagrams for antennas and the relation between antenna theories*, Tech. Report No. 46, Cruft Laboratory, Harvard Univ., 1948.

[15]E. Hallén, *Traveling waves and unsymmetrically fed antennas*, Tech. Report No. 49, Cruft Laboratory, Harvard Univ., 1948.

that such a choice of a trial function is justified, it appears from the foregoing discussion that consideration must be given to *both* Eqs. (11) and (19) in making a final choice of ψ. Note that these two expressions are identical at all resonant lengths ($\beta h = \pi/2$, $3\pi/2$, $5\pi/2$, $\cdots$). Also since $\Psi(z)$ is not predominantly real for the longer lengths, the final choice of ψ may best be a complex value.

Back-scattering cross section appears to be a property of the dipole antenna which is particularly sensitive to prediction by theory. As an illustration of its sensitivity to the choice of expansion parameter, Fig. 9 shows the back-scattering cross section at $\beta h = 2\pi$, $\Omega = 10$, for the shorted dipole as a function of the expansion parameter. A factor greater than two exists between the result using the King-Middleton parameter of about 6 at this length, and the Hallén parameter of 10 in the first-order theory.

VIII. Conclusion. In view of the fact that the series solution of the integral equation has been studied, criticized, and modified by many authors since Hallén's first paper in 1938, and since a theory which can be practically computed does not seem to exist which adequately predicts the complete behavior of a simple dipole antenna, it appears perhaps that a new attack on the problem is justified.

It is significant that the results of Van Vleck, et al,[16] for the back-scattering cross section of a shorted dipole agree more closely with experiment than the first-order solutions of Hallén, King and Middleton, or Miss Gray. Such a comparison is made in Fig. 16 of Ref. 1. Hallén's recent solution[15] for the driven dipole may be an improved one from the standpoint of gain. Some attempts have been made to solve the integral equation by variational methods.[17,18] Storer's solution fails for βh greater than $3\pi/2$. Tai removed this difficulty but his first-order values of R_a at the first resonant length are still higher than those of King and Middleton. It may be worthwhile to follow up a suggestion made by Brillouin[19] that the known function and the kernel of the integral equation be expanded in Fourier series with known coefficients, and that the unknown function for the current be likewise expanded with unknown coefficients. Term-by-term integration would then lead to a set of simultaneous equations for determination of the coefficients. No published results of such an approach have come to the author's attention.

APPENDIX

$$\mathrm{Ein}\,(x) = \mathrm{Cin}\,(x) + j\,\mathrm{Si}\,(x)$$

$$\mathrm{Cin}\,(x) = \int_0^x \frac{1 - \cos t}{t}\,dt$$

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt$$

[16]J. H. Van Vleck, F. Bloch and M. Hamermesh, *Theory of radar reflection from wires or thin metallic strips*, J. Appl. Phys. 18, 274 (1947).

[17]J. E. Storer, *Variational solution to the problem of the symmetrical cylindrical antenna*, Tech. Report No. 101, Cruft Laboratory, Harvard Univ., 1950.

[18]C. T. Tai, *A variational solution to the problem of cylindrical antennas*, Tech. Report No. 12, SRI Project No. 188, Stanford Research Institute, 1950.

[19]L. Brillouin, *Antennas for ultra-high frequencies*, Electrical Communication 22, 11 (1944).