

QUARTERLY OF APPLIED MATHEMATICS

Vol. X

January, 1953

No. 4

ON TAYLOR'S HYPOTHESIS AND THE ACCELERATION TERMS IN THE NAVIER-STOKES EQUATIONS*

By

C. C. LIN**

Massachusetts Institute of Technology

1. Introduction. For turbulence produced behind the grid in a wind tunnel, Taylor¹ introduced the assumption that the spatial pattern of turbulent motion is carried past a fixed point by the mean wind speed without any essential change. The assumption has been applied to connect the time spectrum with space correlation, and Taylor found indeed good agreement between theory and experiment.

The success of Taylor's hypothesis in the case of wind tunnel turbulence has led people to use it also for shear flow. This is obviously a tentative step until justification can be found. In wind-tunnel turbulence, the justification of this hypothesis is essentially based upon the following two conditions:

- (a) the main flow is uniform,
- (b) the level of turbulence is low.

One purpose of this paper is to demonstrate this justification explicitly as a basis for a similar discussion for shear flow, in which case neither of these conditions are as well satisfied. It seems plausible that an eddy of considerable size would change considerably by the shearing motion while it is carried past a given point by the main flow. In §6, estimates are made for the limitation, due to this cause, of the eddy size in the boundary layer, to which Taylor's hypothesis can be applied; and it appears that the major part of the shear contributing fluctuations are of such low time frequencies that Taylor's hypothesis cannot be immediately justified for its use in the determination of their space scale.

In preparation for the analysis of Taylor's hypothesis, it is necessary to give an estimate of the orders of magnitude of the various acceleration terms in the equations of motion. Such an estimate also gives us some insight into the nature of turbulence. The analysis is carried out in §§2-5. One main point is to estimate the acceleration due to pressure gradient. This is first done in §2 in a general manner without using Heisenberg's specific hypothesis² of the independence of the various harmonic components of velocity fluctuations. In §§3-4, general analyses are made of the correlation of the various acceleration terms. All the correlations are expressed explicitly in terms of double and triple

Received February 27, 1952.

*This work has been sponsored by the Office of Naval Research and was carried out under project number NR-044-003, entitled "Numerical Analysis and Theoretical Mechanics" at the U. S. Naval Ordnance Laboratory.

**Consultant, U. S. Naval Ordnance Laboratory.

velocity correlations and certain two-point quadruple correlations. In §5, Heisenberg's hypothesis is used in the equivalent form given by Batchelor³ for correlation functions, and his results for pressure correlations are reproduced by a slightly different derivation. In addition, we have calculated the magnitudes of the inertial acceleration and the instantaneous local acceleration. At large Reynolds numbers, they are the dominant terms in the equations of motion, the acceleration due to pressure forces being negligible in comparison. This suggests that, in a sense, the fluid particles are moving with very little interaction among them. Speculations based on this notion should be made cautiously.

2. Pressure fluctuations. The fluctuations of pressure are related to those of velocity through the equations of motion. By making use of the hypothesis of independence of the harmonic components of the velocity fluctuations, Heisenberg obtained certain relations, in the case of isotropic turbulence, connecting the spectrum of the pressure fluctuations with the spectrum for kinetic energy. An equivalent development using correlation functions was later given by Batchelor. In this section, we shall develop two relations regarding pressure fluctuations without making specific assumptions.

The fluctuations of velocity u_i and pressure p satisfy the Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i, \quad (2.1)$$

and the equation of continuity

$$\frac{\partial u_k}{\partial x_k} = 0, \quad (2.2)$$

where ρ is the density of the fluid, ν is the kinematic viscosity coefficient, and x_i 's are the coordinates of a point P . By multiplying (2.1) with the pressure fluctuation p' at another point p' , and taking averages, we obtain

$$\left\langle p' u_i \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{\rho} \left\langle p' \frac{\partial p}{\partial x_i} \right\rangle; \quad (2.3)$$

where we have made use of the relations

$$\left\langle p' \frac{\partial u_i}{\partial t} \right\rangle = 0, \quad \text{and} \quad \langle p' \Delta u_i \rangle = 0 \quad (2.4)$$

since they are solenoidal tensors of the first rank⁴. From (2.3), we see that

$$\left\langle -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} u_i \frac{\partial u_i}{\partial x_j} \right\rangle = \left\langle \left(-\frac{1}{\rho} \frac{\partial p'}{\partial x_i} \right) \left(-\frac{1}{\rho} \frac{\partial p}{\partial x_i} \right) \right\rangle \quad (2.5)$$

(either with or without summation with respect to the index i). We shall introduce the notation

$$\alpha_i = u_j \frac{\partial u_i}{\partial x_j}, \quad \pi_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i}.$$

Then, (2.5) can be written as

$$\langle \pi_i \alpha_i' \rangle = \langle \pi_i \pi_i' \rangle, \quad (2.6)$$

which yields

$$\langle \pi_i \alpha_i \rangle = \langle \pi_i \pi_i \rangle. \quad (2.7)$$

Thus, the correlation of inertial acceleration and the acceleration due to pressure gradient is equal to the mean square of the latter. By Schwarz inequality,

$$\langle \pi_i \alpha_i \rangle^2 \leq \langle \pi_i^2 \rangle \cdot \langle \alpha_i^2 \rangle$$

we have then

$$\langle |\pi_i|^2 \rangle \leq \langle |\alpha_i|^2 \rangle, \quad (2.8)$$

giving an upper bound for the pressure gradient in terms of velocity correlations. It may be expected that

$$\langle |\alpha_i|^2 \rangle \sim \langle u^2 \rangle^2 / \lambda^2$$

so that $\langle |\pi_i|^2 \rangle$ cannot be larger than the order of $\langle (u^2) \rangle^2 / \lambda^2$. Thus, all the terms in (1.1) are limited to this order of magnitude for reasonable large Reynolds numbers, since, in that case, $\langle (\nu \Delta u_i)^2 \rangle$ is definitely of a smaller order of magnitude.

For very large Reynolds numbers, one may get the order of magnitude of $\langle |\pi_i|^2 \rangle$ by extending Kormogoroff's concept to include pressure fluctuations. If we assume that $1/\rho^2 \langle (p - p')^2 \rangle$ depends, for small values of r , only on the velocity scale $v = (\nu \epsilon)^{1/4}$ and the length scale $\eta = (\nu^3/\epsilon)^{1/4}$, where ϵ is the rate of energy dissipation, then

$$\frac{1}{\rho^2} \langle (p - p')^2 \rangle = v^4 F(r/\eta), \quad (2.9)$$

where F is a universal function of its argument. From this, it follows that

$$\frac{1}{\rho^2} \left\langle \left(\frac{\partial p}{\partial x} \right)^2 \right\rangle = \frac{v^4}{\eta^2} F''(0)$$

or

$$\frac{1}{\rho^2} \left\langle \left(\frac{\partial p}{\partial x} \right)^2 \right\rangle = \frac{\langle u^2 \rangle^2 \cdot \text{const.}}{\lambda^2 \cdot R_\lambda}, \quad R_\lambda = \frac{\sqrt{\langle u^2 \rangle} \lambda}{\nu}. \quad (2.10)$$

This is in conformity with the result obtained above. Equation (2.10) has been previously obtained by Heisenberg with a specific value for the constant, but the derivation depends on specific assumptions. Batchelor used an equivalent set of assumptions, but he also pointed out that these assumptions may be questionable for small distances. Since the evaluation of the magnitude of the pressure gradient definitely depends on pressure correlations at small distances, it seems desirable that an independent argument be given.

In the following, we shall examine all the acceleration terms in the equations of motion.

3. Some remarks on isotropic tensors of rank two. We shall first develop a few general relations for isotropic tensors of the second rank, which will be useful for following developments. Robertson⁴ has shown that such a tensor can always be represented by

$$R_{ik} = R_1(r) \frac{\xi_i \xi_k}{r^2} + R_2(r) \delta_{ik}, \quad (3.1)$$

where (ξ_1, ξ_2, ξ_3) are the coordinates, and

$$r^2 = \xi_1^2 + \xi_2^2 + \xi_3^2. \quad (3.2)$$

Thus, there are two scalar functions required for the definition of such a tensor. He has also shown that, for a tensor of vanishing divergence (a solenoidal tensor), there is only one defining scalar function. These results are generalizations of those for velocity correlations given by von Kármán⁵. The general form of such a tensor is

$$Q_{ik} = -\frac{1}{2r} \frac{\partial F}{\partial r} \xi_i \xi_k + \left(F + \frac{r}{2} \frac{\partial F}{\partial r} \right) \delta_{ik}. \quad (3.3)$$

There is another kind of tensor defined by only one scalar function. This is constructed by starting with a single scalar function $P(r)$ and perform the gradient operation twice; thus,

$$P_{ik} = \frac{\partial^2 P(r)}{\partial \xi_i \partial \xi_k}. \quad (3.4)$$

Such a tensor shall be called a *gradient* tensor. By carrying out the calculations indicated in (3.4), we find

$$P_{ik} = \frac{1}{r} \frac{\partial G}{\partial r} \xi_i \xi_k + G \delta_{ik}, \quad G = P'(r)/r. \quad (3.5)$$

By comparing (3.3) and (3.5), it is clear that a solenoidal tensor is a gradient tensor if both the conditions

$$\frac{\partial G}{\partial r} = -\frac{1}{2} \frac{\partial F}{\partial r}, \quad G = F + \frac{r}{2} \frac{\partial F}{\partial r}$$

are satisfied. Then

$$F + r \frac{\partial F}{\partial r} = \text{const.},$$

and hence

$$F = C_1 + \frac{C_2}{r}. \quad (3.6)$$

If we are restricted to tensors finite everywhere, the only tensors which are both a solenoidal tensor and a gradient tensor are constant multiples of the Kronecker delta δ_{ik} .

From the number of scalar functions involved, it seems reasonable to expect that a general correlation tensor of rank two can be expressed as the sum of one gradient part and one solenoidal part. We shall now show that this can be done in a *unique* manner, if all the tensors vanish for $r \rightarrow \infty$.

First, if there were two such decompositions, then

$$R_{ik} = P_{ik}^{(1)} + Q_{ik}^{(1)} = P_{ik}^{(2)} + Q_{ik}^{(2)},$$

and

$$\Delta_{ik} = P_{ik}^{(1)} - P_{ik}^{(2)} = Q_{ik}^{(2)} - Q_{ik}^{(1)}$$

is both a gradient tensor and a solenoidal tensor, since linear combinations of such tensors obviously retain the original properties (cf. (3.3) and (3.5)). Thus, Δ_{ik} can only

be a multiple of $\delta_{i,k}$. However, by letting $r \rightarrow \infty$, it is clear that the constant multiplier must be zero. Hence, the uniqueness of our decomposition is proved.

The decomposition is effected by setting

$$\left. \begin{aligned} R_1 &= r \frac{\partial}{\partial r} \left(G - \frac{F}{2} \right), \\ R_2 &= G + F + \frac{r}{2} \frac{\partial F}{\partial r}, \end{aligned} \right\} \tag{3.7}$$

which yields

$$\left. \begin{aligned} F &= -\frac{2}{3} \left\{ \frac{1}{r^3} \int_r^\infty r^2 (R_1 + 3R_2) dr - \int_r^\infty \frac{R_1}{r} dr \right\}, \\ G &= -\frac{1}{3} \left\{ \frac{1}{r^3} \int_r^\infty r^2 (R_1 + 3R_2) dr + 2 \int_r^\infty \frac{R_1}{r} dr \right\}. \end{aligned} \right\} \tag{3.8}$$

For convenience of reference, some of the properties of gradient and solenoidal tensors are summarized in the following table.

	Gradient tensor P_{ik}	Solenoidal tensor Q_{ik}
General form	$P_{ik} = \frac{1}{r} \frac{\partial G}{\partial r} \xi_i \xi_k + G \delta_{ik}$ $(P' = rG)$	$Q_{ik} = -\frac{1}{2r} \frac{\partial F}{\partial r} \xi_i \xi_k$ $+ \left(F + \frac{r}{2} \frac{\partial F}{\partial r} \right) \delta_{ik}$
Defining scalar (R_{11} for $\xi_i = (r, 0, 0)$)	$P''(r) = G + r \frac{\partial G}{\partial r}$	$F(r)$
Laplacian ΔR_{ik}	$P \rightarrow \Delta P$	$F \rightarrow \frac{\partial^2 F}{\partial r^2} + \frac{4}{r} \frac{\partial F}{\partial r}$
Trace	$\Delta P = 3G + r \frac{\partial G}{\partial r}$	$3F + r \frac{\partial F}{\partial r}$

If P_{ik} and Q_{ik} are the components of the general tensor R_{ik} , then F and G are given by (3.8) and the traces of P_{ik} and Q_{ik} are given by

$$\left. \begin{aligned} \Delta P &= (R_1 + R_2) - 2 \int_r^\infty \frac{R_1}{r} dr, \\ \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 F) &= +2R_2 + 2 \int_r^\infty \frac{R_1}{r} dr. \end{aligned} \right\} \tag{3.9}$$

4. Correlation of the acceleration terms in the equations of motion. The equations of motion

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i \quad (4.1)$$

contain the four types of terms

$$a_i = \frac{\partial u_i}{\partial t}, \quad (4.2)$$

$$\alpha_i = u_j \frac{\partial u_i}{\partial x_j}, \quad (4.3)$$

$$\pi_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad (4.4)$$

$$\mu_i = \nu \Delta u_i. \quad (4.5)$$

Among them, sixteen two-point correlations can be formed. By symmetry, only ten of them are distinct. Two of them, namely

$$\langle \pi_i a'_k \rangle \quad \text{and} \quad \langle \pi_i \mu'_k \rangle,$$

are identically zero, since they can be expressed in terms of the solenoidal tensors $\langle p a'_k \rangle$ and $\langle p \mu'_k \rangle$ which are identically zero themselves. We are, therefore, left with eight independent tensors.

By forming correlations of (4.1) at $P(x_i)$ with each of the four vectors (4.2) – (4.5) at the other point $P'(x'_i)$, we obtain four relations among the eight tensors, thus reducing the number of independent tensors to four. For systematic analysis, we may choose these to be $\langle a_i a'_k \rangle$, $\langle \alpha_i \alpha'_k \rangle$, $\langle \pi_i \pi'_k \rangle$ and $\langle \mu_i \mu'_k \rangle$. The other tensors are then expressible in terms of these as follows:

$$\langle a_i \alpha'_k \rangle = \langle \alpha_i a'_k \rangle = \frac{1}{2} \{ -\langle a_i a'_k \rangle + \langle \mu_i \mu'_k \rangle - V_{ik} \}, \quad (4.6)$$

$$\langle a_i \pi'_k \rangle = \langle \pi_i a'_k \rangle = 0, \quad (4.7)$$

$$\langle a_i \mu'_k \rangle = \langle \mu_i a'_k \rangle = \frac{1}{2} \{ \langle a_i a'_k \rangle + \langle \mu_i \mu'_k \rangle - V_{ik} \}, \quad (4.8)$$

$$\langle \alpha_i \pi'_k \rangle = \langle \pi_i \alpha'_k \rangle = \langle \pi_i \pi'_k \rangle, \quad (4.9)$$

$$\langle \alpha_i \mu'_k \rangle = \langle \mu_i \alpha'_k \rangle = \frac{1}{2} \{ -\langle a_i a'_k \rangle + \langle \mu_i \mu'_k \rangle + V_{ik} \}, \quad (4.10)$$

$$\langle \pi_i \mu'_k \rangle = \langle \mu_i \pi'_k \rangle = 0, \quad (4.11)$$

where V_{ik} is given by

$$\langle \alpha_i \alpha'_k \rangle = \langle \pi_i \pi'_k \rangle + V_{ik}. \quad (4.12)$$

We notice that, from (4.6), V_{ik} is a tensor of vanishing divergence. On the other hand $\langle \pi_i \pi'_k \rangle$ is obviously a gradient tensor. Hence (4.12) is the decomposition of $\langle \alpha_i \alpha'_k \rangle$ into its component parts, and is, therefore, equivalent to *two* relations. These are given by (3.8) of the last section, with the correspondence given by the following relations ($\xi_i = x'_i - x_i$):

$$\left. \begin{aligned} P(r) &= \langle pp' \rangle, \\ P_{ik} &= \rho^2 \langle \pi_i \pi'_k \rangle, \\ Q_{ik} &= \rho^2 V_{ik}, \\ R_{ik} &= \rho^2 \langle \alpha_i \alpha'_k \rangle. \end{aligned} \right\} \quad (4.13)$$

Thus, $\langle \pi_i \pi'_k \rangle$ and V_{ik} are both expressible in terms of $\langle \alpha_i \alpha'_k \rangle$. The independent tensors are then $\langle a_i a'_k \rangle$, $\langle \alpha_i \alpha'_i \rangle$ and $\langle \mu_i \mu'_k \rangle$.

Expression in terms of velocity correlations

We shall now show that all the above correlations can be expressed in terms of velocity correlations. First, let us note that,

$$\langle \mu_i \mu'_k \rangle = \nu^2 \Delta \Delta \langle u_i u'_k \rangle, \tag{4.14}$$

$$\langle \alpha_i \mu'_k \rangle = -\nu \Delta \frac{\partial}{\partial \xi_i} \langle u_i u_j u'_k \rangle, \tag{4.15}$$

$$\langle \alpha_i \alpha'_k \rangle = -\frac{\partial^2}{\partial \xi_i \partial \xi_l} \langle u_i u_j u'_k u'_l \rangle. \tag{4.16}$$

As mentioned above, (4.12) enables us to express both $\langle \pi_i \pi'_k \rangle$ and V_{ik} in terms of $\langle \alpha_i \alpha'_k \rangle$. Furthermore, (4.10) and (4.12) give

$$\langle a_i a'_k \rangle = \langle \mu_i \mu'_k \rangle - 2\langle \alpha_i \mu'_k \rangle + \langle \alpha_i \alpha'_k \rangle - \langle \pi_i \pi'_k \rangle \tag{4.17}$$

allowing $\langle a_i a'_k \rangle$ to be expressible in terms of correlations of velocity components at a given instant. The other equations in (4.6) – (4.10) then give all the other correlations. Thus, there are altogether *four* scalar functions involved in expressing all the correlations of accelerations; two of them are the usual double and triple velocity correlations introduced by (4.14) and (4.15), and the other two relate to quadruple velocity correlations, cf. (4.16).

Magnitude of various acceleration terms.

By contraction and putting $\xi_i = 0$, into (4.9), (4.14), (4.16) and (4.17), we may obtain the various quantities entering into the consideration of the magnitude of the acceleration terms in (3.1). We obtain

$$\langle | \pi_k |^2 \rangle = \langle \alpha_k \pi_k \rangle \leq \langle | \alpha_k |^2 \rangle, \tag{4.18}$$

$$\langle | \mu_k |^2 \rangle = 35\nu^2 \langle u^2 \rangle f_0^{iv}, \tag{4.19}$$

$$\langle \alpha_k \mu_k \rangle = +35\nu \langle u^2 \rangle^{3/2} h_0''', \tag{4.20}$$

$$\langle | \alpha_k |^2 \rangle = - \left[\frac{\partial^2}{\partial \xi_i \partial \xi_l} \langle u_i u_j u'_k u'_l \rangle \right]_{\xi=0}, \tag{4.21}$$

$$\langle | a_k |^2 \rangle = 35\nu^2 \langle u^2 \rangle^2 f_0^{iv} - 70\nu \langle u^2 \rangle^{3/2} h_0''' + \langle | \alpha_k |^2 \rangle - \langle | \pi_k |^2 \rangle, \tag{4.22}$$

where $f(r)$ and $h(r)$ are the double and triple correlation functions introduced by von Kármán and Howarth, and the subscript zero denotes evaluation at $r = 0$. The second relation in (4.18) is obtained by Schwarz's inequality, as noted in §1.

It is quite clear from (4.22) that for large $R_\lambda = u^\lambda/\nu$, the mean square value $\langle | a_i |^2 \rangle$ of the magnitude of a_k is at most of the order of $\langle | \alpha_k |^2 \rangle$, which may be estimated to be of the order of $\langle u^2 \rangle^2/\lambda^2$. To make a closer analysis, one has to introduce suitable approximations for $\langle | \alpha_k |^2 \rangle$ and $\langle | \pi_k |^2 \rangle$. This will be carried out in the next section. It is convenient to put the above relations in dimensionless forms. For example, (4.22) may be written as

$$\frac{\langle | a_k |^2 \rangle}{\langle u^2 \rangle^2/\lambda^2} = \frac{35}{R_\lambda^2} (G - R_\lambda S) + \frac{\langle | \alpha_k |^2 \rangle}{\langle u^2 \rangle^2/\lambda^2} - \frac{\langle | \pi_k |^2 \rangle}{\langle u^2 \rangle^2/\lambda^2}, \tag{4.23}$$

where

$$G = f_0^i \lambda^4, \quad S = 2h_0'' \lambda^3, \tag{4.24}$$

and (4.19) may be written as

$$\frac{\langle |\mu_k|^2 \rangle}{\langle u^2 \rangle^2 / \lambda^2} = \frac{35G}{R_\lambda^2}. \tag{4.25}$$

5. Hypothesis for reducing the quadruple correlations. The above relations are all exact. To get more useful results, we shall make use of the assumption³

$$\langle u_i u_j u'_k u'_l \rangle = \langle u_i u'_k \rangle \langle u_j u'_l \rangle + \langle u_i u'_l \rangle \langle u_j u'_k \rangle + \langle u_i u_j \rangle \langle u'_k u'_l \rangle. \tag{5.1}$$

Then $\langle \alpha_i \alpha'_k \rangle$ can be expressed in terms of a single double correlation function. The total number of scalars involved would reduce to two, say $f(r)$ and $h(r)$. In fact, calculations* show that

$$\begin{aligned} \langle u^2 \rangle^{-2} \langle \alpha_i \alpha'_k \rangle &= (4sf f''' + 14ff'' - 16f'^2)(\xi_i \xi_k - s \delta_{ik}) \\ &\quad - (4sf f'' + 14sf'^2 + 10ff') \delta_{ik} \end{aligned} \tag{5.2}$$

where

$$s = r^2, \quad f' = \frac{df}{ds}, \quad \text{etc.} \tag{5.3}$$

Then

$$\langle |\alpha_k|^2 \rangle = 15 \langle u^2 \rangle^2 / \lambda^2. \tag{5.4}$$

By (3.9), we may obtain ΔP from (5.1). Thus,

$$\langle \rho \langle u^2 \rangle \rangle^{-2} \Delta P(r) = 24 \int_0^s f'^2(s) ds + 16sf'(s), \tag{5.5}$$

which gives

$$\langle |\nabla p|^2 \rangle = 24 \langle \rho \langle u^2 \rangle \rangle^2 \int_0^\infty f'^2(s) ds = 12 \langle \rho u^2 \rangle^2 \int_0^\infty \left(\frac{df}{dy} \right)^2 \frac{dy}{y}. \tag{5.6}$$

This formula for the pressure gradient was first obtained by Batchelor.

The magnitude of the other acceleration terms can then be easily obtained. We list below all these relations in dimensionless form:

$$\frac{\left\langle \left| u_i \frac{\partial u_i}{\partial x_i} \right|^2 \right\rangle}{\langle u^2 \rangle^2 / \lambda^2} = 15, \tag{5.7}$$

$$\frac{\langle |\nabla p|^2 \rangle}{\langle \rho u^2 \rangle^2 / \lambda^2} = 3 \left(\frac{\lambda}{\lambda_p} \right)^2 = 12\lambda^2 \int_0^\infty \left(\frac{df}{dy} \right)^2 \frac{dy}{y}, \tag{5.8}$$

$$\frac{\left\langle \left| \frac{\partial u_i}{\partial t} \right|^2 \right\rangle}{\langle u^2 \rangle^2 / \lambda^2} = \frac{35}{R_\lambda^2} (G - R_\lambda S) + 15 - 12\lambda^2 \int_0^\infty \left(\frac{df}{dy} \right)^2 \frac{dy}{y}. \tag{5.9}$$

$$\langle |A_i|^2 \rangle / \{ \langle u^2 \rangle^2 / \lambda^2 \} = 12\lambda^2 \int_0^\infty \left(\frac{df}{dy} \right)^2 \frac{dy}{y} + \frac{35}{R_\lambda^2} (G - R_\lambda S) \tag{5.10}$$

*it has been found convenient to introduce $s = r^2$ as the variable in the process of calculation.

where A_i is the total acceleration

$$A_i = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}. \quad (5.11)$$

The symbol λ_p is defined by (5.8) and has been estimated by Batchelor for various Reynolds numbers. In particular, for very large Reynolds numbers, he found

$$\lambda_p/\lambda = 0.11R_\lambda^{1/2}. \quad (5.12)$$

This is in agreement with the general discussion given in §1, and shows that the pressure gradient has a magnitude much smaller than the inertial acceleration.

6. Taylor's hypothesis. The above results will now be used to examine Taylor's hypothesis. Consider first the case of a hot wire used to measure wind tunnel turbulence behind a grid. To the extent that the field may be regarded as homogeneous and isotropic, we may consider an observer moving with the mean wind speed U and reduce the problem to that of a registering instrument carried at a velocity $-U$ through a field of homogeneous isotropic turbulence at rest. Then, the instrument registers $u(x + Ut, t)$, and

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x}$$

is the time rate of change as observed by the instrument. Taylor's hypothesis that the instrument registers the space variation is essentially based on the argument that

$$\left\langle \left(\frac{\partial u}{\partial t} \right)^2 \right\rangle \ll U^2 \left\langle \left(\frac{\partial u}{\partial x} \right)^2 \right\rangle, \quad (6.2)$$

It is seen from the above analysis that this is indeed the case provided

$$\frac{\langle u^2 \rangle}{U^2} \ll 1.$$

To be more precise, (5.9) gives for large Reynolds numbers,

$$\left\langle \left(\frac{\partial u}{\partial t} \right)^2 \right\rangle : U^2 \left\langle \left(\frac{\partial u}{\partial x} \right)^2 \right\rangle = 5 \frac{\langle u^2 \rangle}{U^2} \quad (6.3)$$

This may then serve as a guide for deciding on the accuracy of Taylor's hypothesis. Note that the number 5 is obtained by using the specific hypothesis in §5, but it should be correct in its order of magnitude.

Similar remarks apply to derivatives of higher orders.

Another way of looking at Taylor's hypothesis is to say that the main terms in the equations of motion

$$\frac{\partial u_i}{\partial t} + U \frac{\partial u_i}{\partial x} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i \quad (6.4)$$

are the first two, the others being much smaller in comparison. This is seen from the rough argument that the pressure terms are of the same order or smaller than the convective terms $u_j \partial u_i / \partial x_j$ as justified by the previous analysis. Indeed, the pressure terms are often much smaller than indicated by such a crude argument.

For the case of shear flow, it is difficult to make an exact analysis, and one may first proceed with the kind of rough arguments just given, with its plausibility supported by

its correctness in the case of uniform flow. However, the pressure fluctuations are not only related to the quadratic terms in the fluctuations but also to the convection of momentum of the mean motion due to turbulent fluctuations. Thus, if $U(y)$ is the basic parallel flow, in the equations of motion

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{dU}{dy} + u_i \frac{\partial u}{\partial x_i} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \Delta u, \\ \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + u_i \frac{\partial v}{\partial x_i} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \Delta v, \\ \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + u_i \frac{\partial w}{\partial x_i} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta w, \end{aligned} \right\} \quad (6.5)$$

there is the term $v dU/dy$. If Taylor's hypothesis were true. Then $u = f(x - Ut)$, $v = g(x - Ut)$, $w = h(x - Ut)$, and we have approximately, for large R_λ and small turbulence levels,

$$\left. \begin{aligned} \frac{\partial p}{\partial x} &= -\rho v \frac{dU}{dy}, \\ \frac{\partial p}{\partial y} &= 0, \\ \frac{\partial p}{\partial z} &= 0. \end{aligned} \right\} \quad (6.6)$$

Hence,

$$p = -\rho \frac{dU}{dy} G(x - Ut), \quad \text{where} \quad G'(\xi) = g(\xi),$$

and the second equation of (6.6) gives

$$\frac{\partial p}{\partial y} = -\rho \frac{d^2 U}{dy^2} G(\xi) + \rho \left(\frac{dU}{dy} \right)^2 g(\xi) \frac{x - \xi}{U} = 0$$

this is generally true only when

$$\frac{dU}{dy} = 0, \quad \frac{d^2 U}{dy^2} = 0,$$

i.e., when the main flow is uniform. There is, therefore, no general justification of extending Taylor's hypothesis to the case of shear flow.

One may still attempt to justify it on the basis that

$$\left| U \frac{\partial u}{\partial x} \right| \gg \left| v \frac{dU}{dy} \right|.$$

Then it may be expected to hold only for components of wave length k such that

$$kU \gg \frac{dU}{dy} \quad \text{or} \quad ky \gg \frac{y}{U} \frac{dU}{dy}.$$

The latter form is useful when the experimental data are presented in scales of $\log y$. For boundary layers, experimental values at $y/\delta = 0.2$ (Townsend⁶) give roughly

$$ky \gg 0.9 \quad \text{or} \quad k\delta \gg 4.5.$$

In terms of the wave-length Λ of the disturbance, this means that

$$\frac{\Lambda}{\delta} \ll 0.7.$$

Thus, the wave-length must be much less than the boundary layer thickness in order that Taylor's hypothesis may apply.

For smaller values of y/δ , the restriction is even more severe. For larger values y/δ , the condition

$$ky \gg 1$$

may be taken as a rough approximation, until one comes to the region where intermittency becomes appreciable ($\xi = 0.02$ in Townsend's experiment). Before this point is reached, and with $y \geq 0.2 \delta$, $y(dU/dy)$ and U both increase slightly with y , making the resultant change in the quantity $(y/U) (dU/dy)$ fairly small.

Thus, at $y/\delta = 0.6$, we must have

$$k\delta \gg 1.5,$$

or

$$\frac{\Lambda}{\delta} \ll 4.$$

This means that for harmonic components with wave-lengths of the order of magnitude of the boundary layer thickness, Taylor's assumption is barely applicable at $y/\delta = 0.6$.

Klebanoff and Diehl⁷ presented their results in terms of nL_x/U , where n is the time frequency in cycles, and $L_x = 0.4 \delta$. The above condition becomes then

$$\frac{nL_x}{U} \gg 0.1.$$

With reference to their Figure 25, it is seen that this condition is satisfied from the $n^{-5/3}$ range upwards. Shear-contributing fluctuations, however, do not satisfy this requirement.

It should be noted, however, that the wave-length U/n that may be assigned to the $n^{-5/3}$ range is of the order of magnitude of the thickness of the boundary layer ($\delta/2$ to δ). This is rather large for the application of Kolmogoroff's theory. The possibility is not to be excluded that the disturbances are actually propagating downstream at a lower speed analogous to oscillations in a laminar boundary layer.

Referring to (6.3), it is seen that for the observed level of turbulence in the boundary layer, the non-linear terms does not cause much inaccuracy in the application of Taylor's hypothesis. This is contrary to the conjecture that might be based on the fact that the quadratic terms are important in the phenomena of turbulent motion.

REFERENCES

1. G. I. Taylor, Proc. Roy. Soc. A **164**, 476-490 (1938).
2. W. Heisenberg, Z. f. Phys. **124**, 628 (1948).
3. G. K. Batchelor, Proc. Camb. Phil. Soc. **47**, 359-374 (1951).
4. H. P. Robertson, Proc. Camb. Phil. Soc. **36**, 209-223 (1940).
5. Th. von Kármán, J. Aero. Sci. **4**, 131-138 (1937); Th. von Kármán and L. Howarth, Proc. Roy. Soc. Lond. A **164**, 192-215 (1938).
6. A. A. Townsend, Proc. Camb. Phil. Soc. **47**, 375-395, (1951).
7. P. S. Klebanoff and Z. W. Diehl, NACA Tech. Note 2475 (October, 1951).