1. Introduction and summary. As an example of their “variational method”, LEVINE and SCHWINGER [1] investigated a boundary value problem which arises from the diffraction of a plane scalar (acoustical) wave by a plane screen with a circular aperture. It is equivalent to the problem of finding the field of a freely vibrating circular disk. A full discussion of the physical problems was given by Bouwkamp [2]. Let $z, \rho, \theta$ be cylindrical coordinates and let $z = 0$ be the plane occupied by the screen. Let $z = 0, 0 \leq \rho < a$ define the aperture (or the vibrating disk). The diffracted field is given by a function $u$ which satisfies $\nabla^2 u + k^2 u = 0$ (with a constant $k$) everywhere except for $z = 0$ and at infinity satisfies a Sommerfeld radiation condition. For $z = 0$, $u$ must satisfy the “mixed” boundary conditions $u = 0$ for $\rho > a$ and $\partial u / \partial z = v_0$ with a given constant value $v_0$ for $0 \leq \rho < a$. These conditions determine $u$ uniquely. For $z = 0, 0 \leq \rho < a$, $u = \Phi(\rho)$ becomes a function of $\rho$ only, and if $\Phi(\rho)$ is known or even if only $C_0 \Phi(\rho)$ with an undetermined constant factor $C_0$ is known, $u$ can be determined everywhere; see formulas (A.1), (A.2), (A.3) in [1].

LEVINE and Schweringer [1] show that the ratio of the energy transmitted through the aperture to the energy incident on the aperture is the imaginary part of the complex transmission coefficient $T^*$, which is a quotient of two integrals involving $\Phi(\rho)$ quadratically. As a functional of $\Phi(\rho)$, $T^*$ becomes stationary for the correct function $\Phi$ which determines $u$. Levine and Schweringer find approximate values for $T^*$ by expanding first $\Phi(\rho)$ in an infinite series of auxiliary functions (see 3.1 and 3.2) with coefficients $D_m$. Then $T^*$ becomes a linear form in the $D_m$ (see 3.10), and the unknowns $D_m$ are determined by an inhomogeneous system of infinitely many linear equations with a coefficient matrix $L$ (see 3.4, 3.5). In [1], these equations are solved “section wise”, using the first $l = 1, 2, 3, \cdots$ equations to determine the first $l$ unknowns. All quantities $D_m$, $T^*$, $L$ are power series in $\beta = ka/2$, and Levine and Schweringer compute the first coefficients of the expansion of $T^*$ in a power series in $\beta$ which were determined independently by Bouwkamp [2], who used spheroidal wave functions.

It will be shown that the algebraic properties of the matrix $L$ make it possible not only to find approximate values for $T^*$ as in [1] but also to determine $\Phi(\rho)$. This is due to the fact that $L$ factorizes in a product $L^{(0)} S$, where $L^{(0)}$ is the matrix for the static case $k = 0$ and where $S$ can be inverted by solving finite recurrence relations. The details are stated in lemma 1 and theorem 1 of section 3. Lemma 2 gives additional algebraic relations. Problems of convergence and uniqueness are settled in section 5. These depend largely on an investigation of the properties of $L^{(0)}$ which is carried through in section 4. There it is shown that in the limiting cases $k = 0$ and $k = \infty$ the matrices $L^{(0)}$ and $L^{(\infty)}$ of the linear equations also arise from a problem of moments. This also makes it possible to prove that the variational method for the calculation of the transmission
coefficient will work even for \( k = \infty \) where the linear equations for the \( D_m \) do not have any solution at all.

2. Notations. The elements of (infinite) matrices are denoted by subscripts \( n, m = 0, 1, 2, \ldots \) where \( n \) denotes the rows and \( m \) denotes the columns. A vector with components \( x_m \) is denoted by \( \{x_m\} \). We also use the notations

\[
(a)_n = \Gamma(a + n)/\Gamma(a) = a(a + 1) \cdots (a + n - 1); \quad (a)_0 = 1,
\]

\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,
\]

where \( \Gamma \) denotes the gamma function and \( F \) denotes the hypergeometric series. For results needed here see Whittaker and Watson [3] and Bailey [4].

3. Algebraic properties of the linear equations. Let

\[
\Phi(p) = -\frac{1}{2} aC_0 \sum_{n=0}^{\infty} x_m (1 - \rho^2/a^2)^{m+1/2}
\]

be the expansion of the field \( \Phi(p) \) in the aperture in terms of powers of \( 1 - \rho^2/a^2 \). Here \( C_0 \) denotes an undetermined constant and

\[
-\frac{1}{2} a x_m = D_m
\]

where the \( D_m \) are the unknowns used by Levine and Schwinger [1]. The linear equations for the \( x_m \) as obtained from the variational method can be written as follows:

Let \( p, q = 0, 1, 2, \ldots \) and let \( L^{(2p)}, L^{(2q+3)} \) be infinite matrices with elements \( l_{n,m}^{(2p)}, l_{n,m}^{(2q+3)} \) defined by

\[
l_{n,m}^{(2p)} = (-1)^p n^{1/2} A(n, m, p)/B(n, m, p),
\]

\[
l_{n,m}^{(2q+3)} = i(-1)^q n^{1/2} A(n, m, q + 3/2)/B(n, m, q + 3/2),
\]

where, for any values of \( n, m, t \)

\[
A(n, m, t) = \Gamma(n + 3/2)\Gamma(m + 3/2)\Gamma(n + m + 2t + 1),
\]

\[
B(n, m, t) = 4\Gamma(t + 1)\Gamma(n + t + 1)\Gamma(m + t + 1)\Gamma(n + m + t + 5/2).
\]

Let \( L \) be the matrix

\[
L = \sum_{p=0}^{\infty} \beta^{2p} L^{(2p)} + \sum_{q=0}^{\infty} \beta^{(2q+3)} L^{(2q+3)},
\]

the general element \( l_{n,m} = l_{n,m}(\beta) \) of which is a power series in \( \beta = \frac{1}{2}ka \). Then

\[
\sum_{n=0}^{\infty} l_{n,m} x_m = (n + 3/2)^{-1}.
\]

Let \( \xi \) denote the vector with the components \( x_m \) and let \( \xi^{(r)}, r = 0, 1, \ldots \) be the vector with the components \( x_m^{(r)} \) where

\[
x_m = \sum_{r=0}^{\infty} \beta^r x_m^{(r)}.
\]

Let \( \eta^{(0)} \) denote the vector with the components \( 1/(m + 3/2) \). Comparing the coefficients of \( \beta^r, r = 0, 1, \ldots, \) on both sides of (3.6) we find

\[
L^{(0)} \xi^{(0)} = \eta^{(0)}, \quad L^{(0)} \xi^{(1)} = 0.
\]
and, for \( r = 2, 3, 4, \ldots \):

\[
L^{(0)}\xi^{(r)} + L^{(2)}\xi^{(r-2)} + \cdots + L^{(r)}\xi^{(0)} = 0. \tag{3.9}
\]

If

\[
T^* = \sum_{m=0}^{\infty} x_m/(m + 3/2), \tag{3.10}
\]

the transmission coefficient \( T \) becomes

\[
T = \beta/2 \text{Im} T^* \tag{3.11}
\]

where \( \text{Im} \) denotes the imaginary part. We shall now show that \( L^{(0)} \) is a common left hand factor of all the matrices \( L^{(2p)}, L^{(2q+3)} \), such that the right hand factor is a bounded matrix.

**Lemma 1.** Let \( p = 1, 2, 3, \ldots \) and \( q = 0, 1, 2, \ldots \), and let \( S^{(2p)} = (s_{n,m}^{(2p)}) \) and \( S^{(2q+3)} = (s_{n,m}^{(2q+3)}) \) be the matrices defined by

\[
s_{n,m}^{(2p)} = 0 \quad \text{if} \quad n > p + m \tag{3.12}
\]

and otherwise

\[
s_{n,m}^{(2p)} = (-1)^p G(n, m, p)/H(n, m, p), \tag{3.13}
\]

\[
s_{n,m}^{(2q+3)} = \gamma(-1)^q G(n, m, q + 3/2)/H(n, m, q + 3/2), \tag{3.14}
\]

where, for any values of \( n, m, t \)

\[
G(n, m, t) = (-t + 3/2)\Gamma(2t - n + m)\Gamma(m + 3/2), \quad H(n, m, t) = \Gamma(t + 1)\Gamma(t)\Gamma(t + m - n + 1)\Gamma(t + m + 3/2)(3/2)^t .
\]

Then

\[
L^{(2p)} = L^{(0)} S^{(2p)}, \quad L^{(2q+3)} = L^{(0)} S^{(2q+3)}. \tag{3.15}
\]

**Proof:** The element in the \( n \)-th row and \( m \)-th column of \( L^{(0)} S^{(2p)} \) is

\[
\frac{\sqrt{\pi}}{4} \frac{(-1)^p}{p!(p-1)!} \frac{\Gamma(n + 3/2)}{\Gamma(m + p + 3/2)} \sum_{r=0}^{p+m} \frac{(n + r)!}{r!} \frac{\Gamma(r + 3/2)}{\Gamma(n + r + 5/2)} \frac{(-p + 3/2)}{(-p + 3/2)} \frac{\Gamma(2p + m - r + 1)}{\Gamma(2p + m - r + 1)}. \tag{3.16}
\]

where, because of (2.1) and simple properties of the Gamma function

\[
\sum_{n,m} = \sum_{r=0}^{p+m} \frac{(n + r)!}{r!} \frac{\Gamma(r + 3/2)}{\Gamma(n + r + 5/2)} \frac{(-p + 3/2)}{(-p + 3/2)} \frac{\Gamma(2p + m - r + 1)}{\Gamma(2p + m - r + 1)} \tag{3.17}
\]

\[
= \frac{n!\Gamma(2p + m)}{(p + m)!\Gamma(n + 5/2)} \sum_{r=0}^{p+m} \frac{(n + 1)!}{r!} \frac{(3/2-p)}{(3/2-p)} \frac{(-p-m)}{(n + 5/2)(1-m-2p)}. \tag{3.18}
\]

The sum in (3.18) can be computed by using Saalschuetz's formula (cf. Bailey [4] for a simple proof) which can be written in the form

\[
\sum_{r=0}^{k} \frac{(a,b,)(-k)}{r!(c),(1+a+b-c-k)} = \frac{(c-a,c-b)}{(c),(c-a-b)}. \tag{3.19}
\]

\((k = 0, 1, 2, \ldots; c \neq 0, -1 - 2, \ldots, -k - 1; 1 + a + b - c \neq 1, 2, \ldots, k)\)
Taking \( a = n + 1, b = -p + 3, c = 5 + r, \) \( h = p + m, \) (3.19) gives for
\[
\sum_{\alpha, \mu} = \frac{n! \Gamma(2\rho + m, \Gamma; 3.2.1) (2\mu + \rho + 1)_{\mu}}{(p + m)! \Gamma(n + 3.2.1) (n + 3.2.1)_{\mu + p} \Gamma(m + p)}.
\]
(3.20)
From (3.20) and (3.16) it follows that \( L^{(2p)} = L^{0} S^{2h} \). The proof of \( L^{(2p+3)} = L^{(0)} S^{(2p+3)} \)
follows by the same method.

The elements of the matrices \( S^{(2p+3)} \) are zero except for those in the first \( p \) rows. This is not true for the \( S^{(2p)} \) but the following lemma shows that \( S^{(2p)} \) is a polynomial in \( S^{(2)} \) apart from right hand factors which are either the identity or of the type of the \( S^{(2p+3)} \).

We have:

**Lemma 2.** Let \( p, t = 1, 2, 3, \ldots \) and let \( R^{(t)} \) be the matrix for which the element in the first row and \( m \)-th column is

\[
(-1)^{t+1} \frac{\Gamma(m + 3.2)(2t + m + 1)}{(t - 1/2)t!(t - 1)! \Gamma(m + 1/2 + 3/2)(t + 1/2)!}.
\]
(3.21)

all other elements of \( R^{(t)} \) being zero. Then

\[
S^{(2)} S^{(2t)} - \frac{t + 1}{1 - 2t} S^{(2t+2)} = R^{(t)},
\]
(3.22)

\[
S^{(2t+2)} = \sum_{\mu=0}^{t} (-2)^{t+1}[-t + 1/2]_{t+1} / (-1 - t)_{t+1} [S^{(2)}] R^{(t-\mu)},
\]
(3.23)

where, for \( \mu = t, R^{(0)} \) denotes \( S^{(2)} \). In general,

\[
S^{(2p)} S^{(2t)} - \frac{(t + p)!}{p!!} \frac{\Gamma(3/2) \Gamma(-t - p + 3/2)}{\Gamma(-p + 3/2) \Gamma(-t + 3/2)} S^{(2p+2t)}
\]
(3.24)
is a matrix in which all elements are zero except those in the first \( p \) rows.

The proof of lemma 2 follows again from Saalschütz's formula. We have now:

**Theorem 1.** If the equations

\[
L^{(0)} \xi^{(0)} = \eta
\]
(3.25)
have a solution, then all the vectors \( \xi^{(m)} \) are determined by \( \xi^{(0)} \) and by the relations \( \xi^{(1)} = 0 \) and the recurrence relations

\[
\xi^{(r)} = -S^{(2)} \xi^{(r-2)} - S^{(3)} \xi^{(r-3)} - \cdots - S^{(r)} \xi^{(0)}.
\]
(3.26)

In the particular case where

\[
\eta = \eta^{(0)} = (2/3, 2/5, 2/7, \ldots),
\]
(3.27)
we have

\[
\xi^{(0)} = (8/\pi, 0, 0, 0, \ldots),
\]
(3.28)
and at most the first \( r + 1 \) components of \( \xi^{(r)} \) are different from zero. \( \xi^{(0)}, \ldots, \xi^{(r)} \) are the solutions of the original system (3.6), if we use the first \( r + 1 \) equations for determining the first \( r + 1 \) unknowns and thereby neglect all terms involving the higher powers of \( \beta \) from the \( r \)-th power onwards. \( \xi^{(0)}, \ldots, \xi^{(r)} \) also determine the exact values of the first \( r + 1 \) coefficients of the expansion of \( T^* \) in powers of \( \beta \).
The proof of theorem 1 follows immediately from lemma 1 and in particular from the fact that the $S^{(2\nu)}$, $S^{(2\nu+3)}$ involve many vanishing elements. The uniqueness of the $\xi^{(r)}$, and the existence of the $x_m$ (at least for sufficiently small values of $\beta$) will be proved in section 5.

4. Limiting cases for the matrix $L$. Let

$$P(t) = \Gamma(t + 3/2)/\Gamma(t + 1), \quad Q(t) = \Gamma(t + 5/2)/\Gamma(t + 1).$$

Then Theorem 1 states that the equations

$$\sum_{m=0}^{\infty} l_{n,m}(\beta)x_m = h_n \quad (n = 0, 1, 2, \cdots)$$

(4.2)

can be solved by formal (i.e. not necessarily convergent) power series in $\beta$ if the equations

$$4L^{(0)}\xi \equiv \left\{ \pi^{1/2}P(n) \sum_{m=0}^{\infty} x_m P(n + m)/Q(n + m) \right\} = \{4h_n\}$$

(4.3)

have a solution $x_m = x_m^{(0)}$. We shall investigate (4.3) together with the limiting case $\beta \to \infty$. Levine and Schwinger [1] have shown that then (4.2) tends towards the system of linear equations

$$L^{(\infty)}\xi \equiv \left\{ \sum_{m=0}^{\infty} x_m/(n + m + 2) \right\} = \mu\{h_n\}, \quad (n = 0, 1, 2, \cdots)$$

(4.4)

where $\mu$ is a constant.

We have to define first the linear space of admissible solutions $x_m$ from the nature of the problem. Since (3.1) is supposed to define the field in the aperture, and since the field cannot have a singularity in the center of the aperture, we must assume that

$$\lim_{\epsilon \to 0} \sum_{m=0}^{\infty} x_m(1 - \epsilon)^m$$

(4.5)

exists. Since the original system (3.6) was set up merely in order to define the transmission coefficient, we shall assume that

$$\sum_{m=0}^{\infty} x_m/(n + 3/2)$$

(4.6)

converges. This implies, that

$$\sum_{m=0}^{\infty} x_m z^m$$

(4.7)

converges for $|z| < 1$ and therefore that the $x_m$ actually define the field in the aperture. Then we prove first:

Lemma 3. If the vector $\xi$ with the components $x_m$ satisfies (4.5) and (4.6), then the operators $L^{(0)}$ and $L^{(\infty)}$ are defined for $\xi$ in the sense that the sums in (4.3), (4.4) converge for $n = 0, 1, 2, \cdots$

Proof: Let $Q(t)$ be defined as in (4.1) and let

$$\tau_m = Q(m)/Q(n + m), \quad \sigma_m = \sum_{r=0}^{m} x_r/(r + 3/2).$$

(4.8)

Then the partial sums of the series in (4.3) are

$$\sum_{r=0}^{m} \tau_r x_r/(r + 3/2) = \sum_{r=0}^{m-1} (\tau_r - \tau_{r+1})\sigma_r + \tau_m \sigma_m$$

(4.9)
where
\[ 2\tau_{r+1} - 2\tau_r = 3nP(r + 1)/[Q(n + r)[n + r + 5/2]]. \] 

(4.10)

Since the \(|\sigma_n|\) are bounded and \(\sum_n |\tau_r - \tau_{r+1}|\) converges, the sums in (4.3) also converge. The proof for the convergence of the sums in (4.4) is even simpler.

**Theorem 2.** If the equations \(L^{(0)}\xi = \{h_n\}\) or \(L^{(\infty)}\xi = \{h^*_n\}\) have a solution \(\xi = |x^{(0)}_m|\) or \(\xi = |x^{(\infty)}_m|\) satisfying (4.5) and (4.6), then the integral equations

\[ \int_0^1 f(v)(1 - v)^{1/2}(1 - vz)^{-1} dv = 4\pi^{-1/2} \sum_{n=0}^\infty z^n h_n n!/(3/2)_n, \]

(4.11)

\[ \int_0^1 f^*(v)(1 - v)^{-1} dv = \sum_{n=0}^\infty h^*_n z^n, \]

(4.12)

have analytic solutions

\[ f(v) = \sum_{m=0}^\infty v^m x^{(0)}_m \Gamma(m + 3/2)/m!, \quad f^*(v) = \sum_{m=0}^\infty x^{(\infty)}_m v^n. \]

(4.13)

The solutions are unique and they also solve the problems of moments

\[ \int_0^1 f(v)(1 - v)^{1/2} v^n dv = 4\pi^{-1/2} h_n n!/(3/2)_n, \quad \int_0^1 f^*(v)v^{n+1} dv = h^*_n. \]

(4.14)

The integrals in (4.11) (4.12) are defined by

\[ \int_0^1 = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon}. \]

(4.15)

Since a formal expansion of the left hand sides of (4.11) and (4.12) leads to the linear equations \(L^{(0)}\xi = \{h_n\}\) and \(L^{(\infty)}\xi = \{h^*_n\}\), it has only to be shown that, under the assumptions made about the \(x_m\), such an expansion is legitimate. It suffices to prove that

\[ \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} f(v)(1 - v)^{1/2} v^n dv = 8\pi^{-1/2} h_n n!\Gamma(n + 3/2) \]

(4.16)

where now \(f(v)\) is defined by (4.13) and \(h_n\) by \(L^{(0)}\xi = \{h_n\}\). Since it follows from the assumption (4.5) about the \(x_m\) that \(f(v)\) converges absolutely and uniformly for \(0 \leq v \leq 1 - \varepsilon\), we may integrate term by term in (4.16). Putting \(Y_m = x^{(0)}_m \Gamma(m + 3/2)/m!\) this gives (with \(v = (1 - \varepsilon)W\))

\[ \sum_{m=0}^\infty Y_m \int_0^{1-\varepsilon} v^{m+3/2}(1 - v)^{1/2} dv \]

(4.17)

\[ = \sum_{m=0}^\infty Y_m (1 - \varepsilon)^{n+m+3} \int_0^1 W^{n+m+3}[1 - (1 - \varepsilon)W]^{1/2} dW \]

\[ = \sum_{m=0}^\infty Y_m (1 - \varepsilon)^{n+m+1}(n + m + 1)^{-1}F(-1/2, n + m + 1, n + m + 2; 1 - \varepsilon) \]

(4.18)

\[ = \sum_{m=0}^\infty Y_m (1 - \varepsilon)^{n+m+1}(n + m + 1)^{-1}F(-1/2, n + m + 1, n + m + 2; 1) \]

(4.19)

\[ + \sum_{m=0}^\infty Y_m (1 - \varepsilon)^{n+m+1}(n + m + 1)^{-1}F(\cdots; 1 - \varepsilon) - F(\cdots; 1). \]
According to Gauss's formula (cf. Whittaker-Watson [3])

\[ F(-1/2, n + m + 1; n + m + 2; 1) = (n + m + 1)! \Gamma(3/2)/\Gamma(n + m + 5/2), \]

and from Abel's lemma and from lemma 3 it follows that

\[ \lim_{\epsilon \to 0} \sum_{m=0}^{\infty} Y_m(1 - \epsilon)^{n+m+1} \Gamma(3/2)(n + m)!/\Gamma(n + m + 5/2) = 4\pi^{-1/2}h_{n!}/(3/2)_n. \]

Now we have to show that the second sum in (4.19) tends towards zero as \( \epsilon \to 0 \). Because of (4.5) it suffices to show that

\[ c_{m,n}(\epsilon) \]

\[ = \Gamma(m + 3/2)[m!]-1[n + m + 1]-1\{F(-1/2, n + m + 1; n + m + 2; 1 - \epsilon) \]

\[ - F(-1/2, n + m + 1; n + m + 2; 1)\}

\[ = \Gamma(m + 3/2)(2m!)-1 \sum_{k=0}^{\infty} [1 - (1 - \epsilon)^{k+1}]

\[ \cdot (1/2)_k/((k + 1)!(n + m + k + 2)) \to 0 \]

as \( \epsilon \to 0 \) uniformly in \( n, m \). We can prove that \( |c_{m,n}(\epsilon)| < \epsilon \) by observing that

\[ 1 - (1 - \epsilon)^{k+1} \leq (k + 1)\epsilon. \]

This and (4.23) gives

\[ |c_{m,n}(\epsilon)| \leq \epsilon \Gamma(m + 3/2)(2m!)-1 \sum_{k=0}^{\infty} (1/2)_k(n + m + k + 2)-1[k!]-1

\[ = \epsilon \Gamma(m + 3/2)(2m!)(n + m + 2)\Gamma(1/2, n + m + 2; n + m + 3; 1)

\[ = \epsilon \Gamma(1/2)\Gamma(m + 3/2)(n + m + 1)!2m!\Gamma(n + m + 5/2)\]

\[ = \epsilon \pi^{1/2} (m + 1)(m + 2) \cdots (m + n + 1)

\[ 2 (m + 3/2)(m + 5/2) \cdots (m + n + 3/2) \leq \epsilon \pi^{1/2}/2 < \epsilon. \]

The uniqueness of the solution follows from

Lemma 4: If \( \sum_{n=0}^{\infty} x_n/(m + 3/2) \) converges, then for \( 0 \leq v < 1 \), \( (1 - v)^{3/2} | f(v) | \) is bounded. The proof follows from summation by parts with the notation (4.8) and from the remark that

\[ \sum_{n=0}^{\infty} \Gamma(m + 5/2) | \sigma_m | v^m/(m + 1)! \leq C[(1 - v)^{-3/2} - 1]v^{-1}, \]

where \( c \) does not depend on \( v \).

Now we can show that (4.3) cannot have a null solution. Because then the difference \( \phi(v) \) of two solutions of (4.11) would satisfy

\[ \int_0^1 \phi(v)(1 - v)^{1/2}v^n \, dv = 0, \quad n = 0, 1, 2, \cdots, \]

and therefore:

\[ \int_0^1 \phi(v)(1 - v)^{1/2}(1 - v)v^n \, dv = 0, \quad n = 0, 1, 2, \cdots \]

But \( \phi(v) (1 - v)^{3/2} \) would be a function continuous in \( 0 \leq v \leq 1 \) according to lemma 4 and therefore (4.27) shows that \( \phi(v)(1 - v)^{3/2} \) would be identically zero.
Conclusions from theorem 1. The equivalence of the equations $L^{(0)}\xi = \{h_m\}$ and $L^{(\omega)}\xi = \{h_m^\omega\}$ to a problem of moments shows that these sets of linear equations are unstable in the following sense: Not only may these equations have no solution at all, but this is certain to happen if we start with a set $\{h_m\}$ of right hand sides for which a solution exists and then change a finite number of the $h_m$ by an amount however small. In this case there does not even exist a continuous function $f(v)$ which satisfies (4.11) or (4.12) with the modified right hand sides.

The integral operators in (4.11), (4.12) are extensions of the linear operators defined by $L^{(0)}$ or $L^{(\omega)}$, since (4.11) or (4.12) may have a continuous solution $f(v)$ which is not analytic. Consequently, a quantity like the transmission coefficient

$$T^* = \int_0^1 f(v)v^{1/2} \, dv = \sum_{m=0}^\infty x_m/(m + 3/2) \quad (4.28)$$

can be defined even in cases where the $x_m$ do not exist. An easy example is offered by the equations

$$\sum_{m=0}^\infty x_m/(n + m + 2) = \mu/(n + 3/2), \quad (n = 0, 1, 2, \cdots) \quad (4.29)$$

which were also investigated by Levine and Schwinger. The corresponding integral equation is

$$\int_0^1 f(v)\psi(1 - v)^{-1} \, dv = \mu \sum_{n=0}^\infty \psi^*/(n + 3/2) = \mu \int_0^1 v^{1/2}/(1 - v)^{-1} \, dv \quad (4.30)$$

which gives

$$f(v) = \mu v^{-1/2}, \quad T^* = \mu. \quad (4.31)$$

In this case no set of $x_m$ satisfying (4.29) can exist. However, it is possible to find sequences of constants $Y_{m}^{(r)}$ such that

$$\sum_{m=0}^\infty Y_{m}^{(r)}(m + n + 2)^{-1} = \psi_n^{(r)} \quad (4.32)$$

exist and

$$\lim_{r \to \infty} \sum_{n=0}^\infty |\psi_n^{(r)} - \mu/(n + 3/2)|^2 = 0, \quad \lim_{r \to \infty} \sum_{m=0}^\infty Y_{m}^{(r)}/(m + 3/2) = \mu. \quad (4.33)$$

For this purpose, we can choose the $Y_{m}^{(r)}$ from

$$\sum_{m=0}^\infty Y_{m}^{(r)}v^m = \sum_{k=0}^r (1 - v)^k(1/2)_k/k! \quad (4.34)$$

The right hand side in (4.34) is a polynomial which approximates $v^{-1/2}$, since it is the $(r + 1)$-th partial sum of $[1 - (1 - v)]^{-1/2}$. Clearly, the $Y_{m}^{(r)} \to \infty$ as $r \to \infty$.

5. Uniqueness and existence of the solution. Once a vector $\xi^{(0)}$ has been determined such that $L^{(0)}\xi^{(0)} = \eta$, where $\eta$ is the vector of the right hand sides in the original equations $L\xi = \eta$, we can determine $\xi$ from

$$M\xi = \xi^{(0)} \quad (5.1)$$
where, for all values of $\beta$, $M$ is defined by

$$M = I + \sum_{p=1}^{\infty} \beta^{2p} S^{(2p)} + \sum_{q=0}^{\infty} \beta^{2q+3} S^{(2q+3)}$$

(5.2)

Here $I$ denotes the identity. We shall call a vector $\xi$ bounded if $\sum |\xi_m|^2 < \infty$ and we shall call a matrix $M$ bounded if there exists a constant $U > 0$ such that for all bounded vectors $\xi$:

$$\xi^* M' M \xi \leq U^2 \sum |\xi_m|^2$$

(5.3)

where $M'$ is the transposed matrix of $M$ and an asterisk denotes the conjugate complex quantity. $U$ is called an upper bound for $M$. It is well known that, if $U_r$ is an upper bound for $S^{(r)} (r = 1, 2, 3 \cdots)$, the matrix $M$ in (5.2) has a bounded inverse $M^{-1}$ if

$$\sum_{r=2}^{\infty} \beta^r U_r < 1$$

(5.4)

$M^{-1}$ can be obtained from a Neumann series. We can use this in order to prove:

**Theorem 3.** Let $L, M, \eta^{(0)}, \xi^{(0)}$ be defined by (3.5), (5.1), (3.27), (3.28). Then $M^{-1}$ exists and is bounded for sufficiently small values of $|\beta| < \beta_0$ and the equations $L\xi = \eta^{(0)}$ have exactly one solution $\xi$ which satisfies (4.5) and (4.6), namely $\xi = M^{-1}\xi^{(0)}$.

**Proof:** Let $V^{(r)}$ be matrices such that

$$\left\{ I + \sum_{r=2}^{\infty} \beta^r S^{(r)} \right\} \left\{ I + \sum_{r=0}^{\infty} \beta^r V^{(r)} \right\} = I.$$  

(5.5)

It is easily seen that the $V^{(r)}$ can be obtained from the $S^{(r)}$ by recurrence formulas. Let $U^{(r)}$ be upper bounds for the $S^{(r)}$ and assume that there exist constants $\Omega_r$ such that

$$\left(1 - \sum_{r=2}^{\infty} \beta^r U_r \right) \left(1 + \sum_{r=2}^{\infty} \beta^r \Omega_r \right) = 1.$$  

(5.6)

This is true if

$$1 - \sum_{r=2}^{\infty} \beta^r U_r$$

(5.7)

is convergent and positive for $0 \leq \beta < \beta_0$. Then it can be shown that $\Omega_r$ is an upper bound for $V^{(r)}$. Since it can also be shown that $x_m$ (the $m$-th component of $\xi = M^{-1}\xi^{(0)}$) is equal to the $m$-th component of

$$\left\{ \sum_{r=2}^{\infty} \beta^r V^{(r)} \right\} \xi^{(0)}$$

(5.8)

it follows that

$$|x_m| \leq \sum_{r=m}^{\infty} \beta^r \Omega_r.$$  

(5.9)

From this it can easily be shown that for $|\beta| < \beta_0$ condition (4.5) for the $x_m$ is satisfied. This proves the existence of $M^{-1}$ and of a bounded $\xi$ satisfying (4.5), (condition (4.6).
is always satisfied for bounded $\xi$ if we can find $U$, which are sufficiently small. We have

**Lemma 4.** The matrices

$$\{S^{(2)}\}, \quad R^{(1)}, \quad S^{(2t+2)}, \quad S^{(2s+3)}$$

have as upper bounds

$$\pi(\pi^2 - 8)^{1/2}/4, \quad 2^{1/2}(\pi^2 - 8)^{1/2}/t!, \quad (2\pi^2 - 16)^{1/2}2^{t+2}(1/2)/(t + 1)!, \quad 2^{s+1}(2\pi^2 - 16)^{1/2}/(q + 1)!$$

The proof is elementary but laborious and will be omitted since the upper bounds are not the best possible ones.

In order to prove the uniqueness of the solution $\xi = M^{-1}\xi^{(0)}$ we observe first that $(M - s)\xi$ is bounded for every $\xi$ merely satisfying (4.5); provided that $s$ is so small that (5.4), with the $U$, from Lemma 4, converges. This can be proved by an elementary investigation of the $S^{(r)}$. Now if there is a $\xi^*$ satisfying (4.5) and (4.6) such that $L\xi^* = 0$, we would have $M\xi^* = \xi^* + \xi$ where $\xi$ is bounded and $L^{(0)}\xi^* + L^{(0)}\xi = 0$. Now it follows from the equivalence of the operator $L^{(0)}$ to the operator of a moment problem (cf. Theorem 2) that $\xi^* + \xi = 0$. Therefore $\xi^*$ is bounded, and since $M^{-1}$ is bounded, $\xi^*$ must be zero since $M\xi^* = \xi^* + \xi = 0$.

No numerical values for the permissible ranges of $s$ are given since it is entirely possible that the inverse $M^{-1}$ exists for all values of $s$. This seems to be indicated by a result of Sommerfeld and Perron [5] who showed that for the related problem of the freely vibrating disc the real part of a resulting set of linear equations can be solved explicitly and without restrictions.

**References**


