ON THE GAPS IN THE SPECTRUM OF THE HILL EQUATION*

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1. Let \( f = f(t) \) be a real-valued, continuous, periodic function of period 1, so that

\[
f(t) \sim \sum_{n=-\infty}^{\infty} c_n \exp(2\pi \text{int}), \quad (c_n = \bar{c}_n),
\]

and consider the Hill equation

\[
x'' + (\lambda + f(t))x = 0, \quad \lambda \text{ real; } \ ' = d/dt.
\]

It is known that (if \( f \neq 0 \)) there exists a sequence of closed intervals \( I_k: \lambda_k \leq \lambda \leq \lambda^* \) (region of stability), where \( \lambda_k < \lambda^* < \lambda_{k+1} \) and \( k = 1, 2, \cdots \), with the property that (2) has some solution \( x \neq 0 \) which is bounded on \( -\infty < t < \infty \) if and only if \( \lambda \) belongs to the closed set \( S = \sum I_k \); cf. [7], p. 14. The complementary set of \( S \) consists of a half-line \( -\infty < \lambda < \lambda_1 \) and the sequence of open intervals \( J_k: \lambda^* < \lambda < \lambda_{k+1} \), \( k = 1, 2, \cdots \).

In several recent papers, various lower bounds for the value \( \lambda^* \), the least point of the set \( S \), in terms of the Fourier coefficients \( c_n \) of \( f(t) \), have been obtained; [11], [5], [3]. The present note will be devoted to the problem of obtaining estimates (upper bounds) of the lengths \( \lambda_{k+1} - \lambda^* \) of the "gaps" \( J_k \) of the set \( S \) in terms of these Fourier coefficients.

It follows from [4], p. 613, that the length of every gap \( J_k \) is surely not greater than

\[
\limsup_{t \to \infty} f(t) - \liminf_{t \to \infty} f(t) \leq 4 \sum_{n=1}^{\infty} |c_n|.
\]

In addition, asymptotic estimates, as \( \lambda^* \to \infty \), for these gaps are known; [2]. In fact, since \( f(t) \) is uniformly continuous on \( 0 \leq t < \infty \), the lengths \( \lambda_{k+1} - \lambda^* \) of the intervals \( J_k \) tend to zero as \( \lambda_{k+1} \to \infty \); loc. cit., p. 850. Furthermore, additional regularity conditions on \( f(t) \) result in more refined estimates. It should be pointed out here that the investigations of [2] related to singular boundary value problems ([8]) on the half-line \( 0 \leq t < \infty \) determined by (2) and a linear, homogeneous boundary condition at \( t = 0 \), and were not confined to the special case that \( f(t) \) be periodic.

Let \( m(\lambda) \), for \( -\infty < \lambda < \infty \), be defined to be the distance from \( \lambda \) to the set \( S \) considered above, so that

\[
m(\lambda) = \text{g.l.b. } |\lambda - \mu|, \quad \mu \text{ in } S.
\]

It will be shown in section 2 below that \( m(\lambda) \) satisfies the inequality

\[
m^2(\lambda) \leq 2 \sum_{n=1}^{\infty} |c_n|^2, \quad \text{provided } \lambda \geq -c_0.
\]

As a consequence of (4) and (5), one readily sees that the lengths \( \lambda_{k+1} - \lambda^* \) of the gaps \( J_k \) satisfy

\[
\lambda_{k+1} - \lambda^* \leq 2\left(2 \sum_{n=1}^{\infty} |c_n|^2\right)^{1/2}, \quad \text{provided } \frac{1}{2} (\lambda_{k+1} + \lambda^*) \geq -c_0.
\]

It will remain undecided whether (6) actually must hold for all gaps \( J_k \), so that the first inequality of (6) would hold without the proviso of the second inequality. In any

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case, it is readily seen that the estimate of (6), when it applies, is an improvement over that of (3), namely 
\[ 4 \sum_{n=1}^{\infty} | c_n |. \]

In this connection, it should be pointed out that Kato [3], by an adaptation of a relation used by Wintner [11], has obtained the inequality
\[ \lambda_1 \geq -c_0 - \left( \frac{1}{8} \right) \sum_{n=1}^{\infty} | c_n |^2, \]
for the least point \( \lambda_1 \) of the set \( S \). (Wintner had previously shown that \( \lambda_1 \geq -c_0 - 2 \cdot \sum_{n=1}^{\infty} | c_n |^2 \).) Consequently, it is easily seen that the first inequality of (6) is surely valid for all gaps \( J_k \) if, for instance, the inequality
\[ \left( \frac{1}{8} \right) \sum_{n=1}^{\infty} | c_n |^2 \leq \left( 2 \sum_{n=1}^{\infty} | c_n |^2 \right)^{1/2} \]
holds. (If one normalizes \( f \) so that its mean value is zero, hence \( c_0 = 0 \), this last inequality is equivalent to \( \int_0^1 f^2 \, dt \leq 256 \).)

Before proceeding to the proof of (5), it can be noted that the first inequality of (5) surely becomes false if the restriction \( \lambda \geq -c_0 \) is dropped. In fact, if \( f(t) = c_0 \), so that (2) becomes the differential equation of the harmonic oscillator, then \( \sum_{n=1}^{\infty} | c_n |^2 = 0 \), and (5) yields the known result that \( m(\lambda) = 0 \) for \( \lambda \geq -c_0 \). However, \( m(\lambda) > 0 \) for \( \lambda < -c_0 \), since \( S \) is the half-line \( -c_0 < \lambda < \infty \). (Wintner had previously shown that \( \lambda_1 \geq -c_0 - 2 \cdot \sum_{n=1}^{\infty} | c_n |^2 \).) Consequently, it is easily seen that the first inequality of (6) is surely valid for all gaps \( J_k \) if, for instance, the inequality
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2. The proof of (5) will depend upon certain results obtained in [6]. Let \( g_1(t), g_2(t), \ldots \), denote a sequence of functions possessing continuous second derivatives on \( 0 \leq t < \infty \), satisfying
\[ g_n(0) = g'_n(0) = 0, \quad (7) \]
and such that \( g_n(t) \to 0 \) uniformly on every finite \( t \)-interval \([0, T] \). Then, if \( g_n \) and \( L(g_n) \) (where \( L(x) = x'' + f x \)) are of class \( L^2([0, \infty)) \), the inequality
\[ m^2(\lambda) \liminf_{n \to \infty} \int_0^\infty g_n^2 \, dt \leq \liminf_{n \to \infty} \int_0^\infty (L(g_n) + \lambda g_n)^2 \, dt \quad (8) \]
holds. This follows readily by a method analogous to that given in [6], p. 580. (It is to be noted that the set \( S \) considered above is identical with the invariant spectrum (Weyl [8], p. 251) associated with the differential equation (2); [9], [1]. Moreover, the investigations of [6] related to the Weyl theory of singular boundary value problems, alluded to in section 1.)

Next, let \( \mu > 0 \), and let \( g_n = y_n h \), where \( h = \sin (\mu t) \) or \( h = \cos (\mu t) \), and the \( y_n = y_n(t) \) are functions possessing continuous second derivatives on \( 0 \leq t < \infty \). In addition, suppose that \( y_n(0) = y'_n(0) = 0 \), so that (7) certainly holds, and that \( y_n \) and \( L(y_n) \) belong to \( L^2(0, \infty) \). Finally, suppose that the \( y_n \) are such that the “\( \liminf \)” appearing on the left side of the inequality (8) can be replaced by “\( \lim \)” for both \( h = \sin (\mu t) \) and \( h = \cos (\mu t) \).

It follows from (8) that
\[ m^2(\lambda) \liminf_{n \to \infty} \int_0^\infty y_n^2 h^2 \, dt \leq \liminf_{n \to \infty} \int_0^\infty ([y_n'' + (\lambda - \mu + f)y_n] h + 2y_n' h')^2 \, dt. \]

If now the \( y_n \) satisfy
\[ \int_0^\infty y_n^2 \, dt \to 0, \quad \int_0^\infty y_n'^2 \, dt \to 0, \quad (n \to \infty), \]
it is seen that

\[ m^2(\lambda) \lim_{n \to \infty} \int_0^\infty y_n^2 \, dt \leq \liminf_{n \to \infty} \int_0^\infty (\lambda - \mu + f)^2 y_n^2 \, dt. \quad (9) \]

Since (9) holds for both functions \( h \), addition of the two corresponding inequality relations yields, in view of the fact that \( \liminf A + \liminf B \leq \liminf (A + B) \), the inequality

\[ m^2(\lambda) \lim_{n \to \infty} \int_0^\infty y_n^2 \, dt \leq \liminf_{n \to \infty} \int_0^\infty (\lambda - \mu + f)^2 y_n^2 \, dt. \quad (10) \]

Let \( T > 0 \) and define the function \( Y_T(t) \) on \( 0 \leq t < \infty \) so that the graph of \( Y_T(t) \) on \( 0 \leq t \leq T \) consists of three line segments joining, in order, the four points \((0, 0), (1, T^{-1}), (T-1, T^{-1}), \) and \((T, 0)\). On \( T < t < \infty \), let \( Y_T(t) = 0 \). It is clear that the corners of this function can be smoothed out so as to obtain a function \( y_T(t) \) satisfying the conditions imposed upon the \( y_n \) above. Furthermore, it is clear that if \( y_n = y_{T_n} \), where \( T = T_n \to \infty \) as \( n \to \infty \), one can arrange that the functions \( y_n \) be such as to make (10) imply

\[ m^2(\lambda) \leq \liminf_{\delta \to \infty} S^{-1} \int_0^\delta (\lambda - \mu + f)^2 \, dt, \quad (\mu \geq 0). \quad (11) \]

(It is clear that the inequality \( \mu \geq 0 \) in (11), and not merely \( \mu > 0 \), can be allowed.) Now suppose that \( \lambda \geq -c_0 \) and choose \( \mu \geq 0 \) so that \( \lambda - \mu = -c_0 \). Then (11), (1), and the Parseval relation yield

\[ m^2(\lambda) \leq \int_0^1 (\lambda - c_0 + f)^2 \, dt = 2 \sum_{n=1}^{\infty} |c_n|^2, \]

so that the relation (5) is now proved.

References