difficult, and probably not possible, to express \( f \) as the sum of a finite number of terms for each of which we can use (3).

**STRESS FUNCTIONS OF MAXWELL AND MORERA**

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**Summary.** A systematic process is devised for the derivation of the stress functions of Maxwell and Morera by the application of the theorem that a vector whose divergence vanishes is solenoidal. Symmetric and anti-symmetric matrices are established, and the elements of these matrices represent the stress functions of Maxwell and Morera respectively.

**Introduction.** The procedure used by Maxwell\(^1\) to derive three stress functions representing six stress components at any point of an isotropic body is similar to that which Morera\(^2\) subsequently applied to the establishment of the corresponding stress functions. This procedure described also by Love\(^3\) consisted in the choice of three stress components; then the substitution of these components into the equilibrium equation led to the remaining three components necessary to satisfy the equations of equilibrium. Later it was discovered by Sir Richard Southwell\(^4\) that Saint Venant’s and Beltrami’s compatibility equations follow from Castigliano’s principle when the strain energy is expressed in terms of Maxwell’s and of Morera’s functions.

**Derivations.** Neglecting the body forces, the equation of equilibrium is

\[
\text{div} \ T = 0, \tag{1}
\]

where \( T \) is the stress tensor. The three equations of equilibrium are obtained by cyclic interchange of \( x, y, z \) in the equation

\[
\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0. \tag{2}
\]

The equations of equilibrium (2) can be written as

\[
\text{div} \ A = 0, \quad \text{div} \ B = 0, \quad \text{div} \ C = 0 \tag{3}
\]

where, because of the theorem that a vector whose divergence vanishes is a solenoidal vector, the following relations exist:

\[
A = \text{curl} \ F, \quad B = \text{curl} \ G, \quad C = \text{curl} \ H. \tag{4}
\]

Consequently, the stress components may be written as follows:

\[
X_x = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad X_y = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad X_z = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y};
\]

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\[ \begin{align*}
Y_x &= \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \quad Y_y = \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \quad Y_z = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}; \\
Z_x &= \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}, \quad Z_y = \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}, \quad Z_z = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}.
\end{align*} \]  

Upon substituting (5) into the equilibrium conditions

\[ X_y = Y_x; \quad X_z = Z_x; \quad Y_z = Z_y \]  

and introducing, for abbreviation, the symbol \( \Gamma = F_1 + G_2 + H_3 \), the following equations are obtained:

\[ \begin{align*}
\frac{\partial F_3}{\partial x} + \frac{\partial G_3}{\partial y} + \frac{\partial (H_3 - \Gamma)}{\partial z} &= 0, \\
\frac{\partial F_2}{\partial x} + \frac{\partial (G_2 - \Gamma)}{\partial y} + \frac{\partial H_2}{\partial z} &= 0, \\
\frac{\partial (F_1 - \Gamma)}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} &= 0.
\end{align*} \]  

Equations (7), in the same manner as Eqs. (2), can be expressed as before: when the divergence of some vector is equal to zero, that vector is solenoidal. Thus the following expressions can be written:

\[ \begin{align*}
F_1 - \Gamma &= \frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z}, \quad G_1 = \frac{\partial U_1}{\partial z} - \frac{\partial U_3}{\partial x}, \quad H_1 = \frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial y}; \\
F_2 &= \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \quad G_2 - \Gamma = \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \quad H_2 = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}; \\
F_3 &= \frac{\partial W_3}{\partial y} - \frac{\partial W_2}{\partial z}, \quad G_3 = \frac{\partial W_1}{\partial z} - \frac{\partial W_3}{\partial x}, \quad H_3 - \Gamma = \frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y}.
\end{align*} \]  

To evaluate the expression \( \Gamma \), the three terms on the diagonal of Eqs. (8) are added:

\[ F_1 + G_2 + H_3 - 3\Gamma = -2\Gamma \]

from which

\[ \Gamma = -\frac{1}{2} \frac{\partial}{\partial x} (W_2 - V_3) - \frac{1}{2} \frac{\partial}{\partial y} (U_3 - W_1) - \frac{1}{2} \frac{\partial}{\partial z} (V_1 - U_2). \]  

Upon substitution of (9) into (8) the following relations are obtained:

\[ \begin{align*}
F_1 &= \frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} - \frac{1}{2} \frac{\partial}{\partial x} (W_2 - V_3) - \frac{1}{2} \frac{\partial}{\partial y} (U_3 - W_1) - \frac{1}{2} \frac{\partial}{\partial z} (V_1 - U_2), \\
G_2 &= \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} - \frac{1}{2} \frac{\partial}{\partial y} (W_2 - V_3) - \frac{1}{2} \frac{\partial}{\partial x} (U_3 - W_1) - \frac{1}{2} \frac{\partial}{\partial z} (V_1 - U_2), \\
H_3 &= \frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y} - \frac{1}{2} \frac{\partial}{\partial z} (W_2 - V_3) - \frac{1}{2} \frac{\partial}{\partial y} (U_3 - W_1) - \frac{1}{2} \frac{\partial}{\partial x} (V_1 - U_2).
\end{align*} \]  

Now, by use of (8) and (10) the stresses in (5) can be determined as follows:

\[ X_x = \frac{\partial^2 W_3}{\partial y^2} + \frac{\partial^2 V_2}{\partial y^2} - \frac{\partial^2}{\partial y \partial z} (W_2 + V_3), \]  

(11)
\[ Y_y = \frac{\partial^2 U_1}{\partial z^2} + \frac{\partial^2 W_3}{\partial x^2} - \frac{\partial^2}{\partial x \partial z} (U_3 + W_1), \]  
\[ Z_z = \frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} - \frac{\partial^2}{\partial x \partial y} (V_1 + U_3); \]

\[ X_y = \frac{1}{2} \frac{\partial^2 U_3}{\partial y \partial z} - \frac{1}{2} \frac{\partial^2 U_3}{\partial z^2} - \frac{\partial^2 W_3}{\partial x \partial y} + \frac{\partial^2 W_3}{\partial x \partial z} + \frac{\partial^2 V_3}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial z \partial y}, \]

\[ X_z = \frac{1}{2} \frac{\partial}{\partial z} \left\{ \begin{array}{c} \frac{\partial}{\partial z} (V_1 + U_3) + \frac{\partial}{\partial x} (W_2 + V_3) + \frac{\partial}{\partial y} (U_3 + W_1) \end{array} \right\} - \frac{\partial^2 W_3}{\partial x \partial y}. \]

In a similar way one obtains

\[ X_z = Z_x = -\frac{1}{2} \frac{\partial^2 (W_1 + U_3)}{\partial y^2} + \frac{1}{2} \frac{\partial^2 (W_2 + V_3)}{\partial x \partial z} + \frac{\partial^2 (V_1 + U_2)}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}, \]

\[ Y_z = Z_y = -\frac{1}{2} \frac{\partial^2 (W_2 + V_3)}{\partial x^2} + \frac{1}{2} \frac{\partial^2 (U_3 + W_1)}{\partial y \partial x} + \frac{\partial^2 (V_1 + U_2)}{\partial x \partial z} - \frac{\partial^2 W_3}{\partial y \partial z}. \]

The normal and tangential stress components expressed by Eqs. (11) to (16) are determined by the elements of a square matrix, which can be written as follows:

\[
\begin{pmatrix}
U_1 & U_2 & U_3 \\
V_1 & V_2 & V_3 \\
W_1 & W_2 & W_3
\end{pmatrix}
\]

when all the elements except those on the diagonal are set equal to zero; i.e., when

\[ U_2 = U_3 = V_1 = V_3 = W_1 = W_2 = 0 \]

and when

\[ U_1 = x_1, \quad V_2 = x_2, \quad W_3 = x_3 \]

there is a symmetric, or diagonal, matrix, whose elements form the Maxwell stress functions. With these elements the stress components are obtained from equations (11) to (16) as follows:

\[ X_x = \frac{\partial^2 X_3}{\partial y^2} + \frac{\partial^2 X_2}{\partial z^2}, \quad Y_y = \frac{\partial^2 X_1}{\partial z^2} + \frac{\partial^2 X_3}{\partial z^2}, \]

\[ Z_z = \frac{\partial^2 X_2}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}, \quad X_y = Y_x = -\frac{\partial^2 X_3}{\partial x \partial y}, \]

\[ Y_z = Z_y = -\frac{\partial^2 X_1}{\partial y \partial z}, \quad X_z = Z_x = -\frac{\partial^2 X_3}{\partial z \partial x}. \]

When the diagonal elements of the Matrix (17) are set equal to zero; i.e., when

\[ U_1 = V_2 = W_3 = 0 \]
and when
\[ W_2 + V_3 = -\psi_1 \]
\[ U_3 + W_1 = -\psi_2 \]
\[ V_1 + U_2 = -\psi_3 \]
the matrix is anti-symmetric, and its elements represent the Morera stress-functions which when substituted into Eqs. (11) to (16) yield the following stress components:

\[
\begin{align*}
Xx &= \frac{\partial^2 \psi_1}{\partial y \partial z}, & Yy &= \frac{\partial^2 \psi_2}{\partial z \partial x}, & Zz &= \frac{\partial^2 \psi_3}{\partial x \partial y}, \\
Xy &= Yx = -\frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_3}{\partial z} \right), \\
Xz &= Zx = -\frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} \right), \\
YZ &= Zy = -\frac{1}{2} \frac{\partial}{\partial z} \left( -\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} \right).
\end{align*}
\]

Thus, the stress tensor expressed by Maxwell and Morera functions was derived by a direct method from diagonal and antisymmetric matrices.

**A REMARK ON INTEGRAL INVARIANTS**

*By H. D. BLOCK (University of Minnesota)*

Let the \(2n\) variables \(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n\) be related to the \(2n\) variables \(Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n\) by a canonical transformation. Let \(\sigma\) be the unit square: \(0 \leq u \leq 1, 0 \leq v \leq 1\), and let \(q_i = f_i(u, v), p_i = g_i(u, v), (i = 1, 2, \ldots, n)\), where \(f_i\) and \(g_i\) have continuous derivatives on \(\sigma\). This induces the relationships \(Q_i = F_i(u, v), P_i = G_i(u, v), (i = 1, 2, \ldots, n)\). Let \(s_i = \bigcup_{(u, v) \in \sigma} (f_i(u, v), g_i(u, v))\) and \(S_i = \bigcup_{(u, v) \in \sigma} (F_i(u, v), G_i(u, v))\), i.e. the maps of \(\sigma\) on the \((q_i, p_i)\) and \((Q_i, P_i)\) planes respectively. Let

\[
\sum_{i=1}^{n} \int_{s_i} dq_i \, dp_i \quad \text{and} \quad \sum_{i=1}^{n} \int_{s_i} dQ_i \, dP_i,
\]

be denoted respectively by

\[
\int \int_{s} \sum_{i=1}^{n} dq_i \, dp_i \quad \text{and} \quad \int \int_{s} \sum_{i=1}^{n} dQ_i \, dP_i.
\]

It is widely\(^1\) believed that under the conditions stated

\[
\int \int_{s} \sum_{i=1}^{n} dq_i \, dp_i = \int \int_{s} \sum_{i=1}^{n} dQ_i \, dP_i.
\]

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