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NOTE ON TAYLOR INSTABILITY*

By GARRETT BIRKHOFF (*Harvard University*)

1. Qualitative discussion. In a well-known paper [3], Sir Geoffrey Taylor has discussed the stability under normal acceleration of a plane interface separating two fluids of different density. His main conclusion (reached several years earlier) is now classic: the interface is *unstable when the light fluid is accelerated towards the dense fluid* and (presumably) stable when the reverse holds. This conclusion has important applications to gas-filled underwater explosion bubbles.

Its applicability to small vapor-filled cavities is however less clear. The stabilizing role of surface tension is known[1] to be important, and Binnie [1] has suggested that surface tension may even be sufficient to compensate for Taylor instability.

The purpose of this note is to show that, in spite of the fact that the denser liquid is being accelerated towards the lighter vapor, *collapsing bubbles are unstable*, and that this result is unaffected by surface tension (though it may be affected by viscosity or thermodynamic considerations). The proof of this fact depends on a consideration of the stability of differential equations near regular singular points: the instability is algebraic, and not of the exponential type usually considered.

2. Negative damping. The formulas underlying the perturbation theory of collapsing spherical cavities are easily found[2]. Let the cavity radius $b(t)$ be given as a function of time, and the interface expressed in spherical coordinates in terms of Legendre polynomials by

$$r = b(t) + \sum_{h=1}^{\infty} b_h(t) P_h(\cos \phi). \tag{1}$$

Supposing the $b_h(t)$ small, and neglecting gravity and surface tension, the condition of constant internal pressure gives†

$$b b_h^{\cdot\cdot} + 3 b^{\cdot} b_h^{\cdot} - (h - 1) b^{\cdot\cdot} b_h = 0. \tag{2}$$

The same formula applies to any surface harmonic of order h.

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[1]This was observed independently in 1951 by R. H. Pennington and R. Bellman at Princeton, and by R. L. Ingraham and the author, but not published.

[2]Formulas (1)-(2) were derived by W. G. Penney and A. T. Price, British Report SW-27 (1942); see R. H. Cole, *Underwater explosions*, Princeton, 1948, p. 311. See also [1, Formulas (2)-(3)].

†For typographical reasons, the dots indicating time derivatives are set as superscripts.

The facts about stability are suggested by the ordinary stability test. Near $b = 0$, $b^{\cdot\cdot} < 0$, and so the coefficient of b_h in (2) is positive—i.e., we have "Taylor stability" in the usual sense[3]. However, the coefficient of $b_h^{\cdot}$ is negative in the collapse phase, and so we have "negative damping". This suggests that vapor-filled bubbles are unstable during collapse.

However, it is more satisfactory to get quantitative results; this we will now do. As a by-product, it will appear that the ordinary stability test is not conclusive (see Theorem 2).

3. Asymptotic analysis. Since the kinetic energy is proportional to $b^3(b^{\cdot})^2$, [2, p. 122, Eq. (6)], which approaches a constant value as $b \downarrow 0$, evidently $b^{1.5}db$ and dt are asymptotically proportional to each other, whence $t \propto b^{2.5}$ and $b \propto t^{0.4}$. Substituting in (2), we get the asymptotic equation

$$b_h^{\cdot\cdot} + \frac{1\cdot 2}{t} b_h^{\cdot} + \frac{.24(h-1)}{t^2} b_h = 0, \qquad \text{near} \qquad b = 0. \tag{3}$$

The sign of the coefficient of $b_h^{\cdot}$ depends on whether the expansion or collapse phase is involved (i.e., on whether $t > 0$, $b^{\cdot} > 0$ or $b^{\cdot} < 0$, $t < 0$).

Equation (3) has a regular singular point[4] at $t = 0$; the indicial equation, corresponding to possible exponents in asymptotic complex solutions

$$b_h = t^{\alpha}(c_0 + c_1 t + c_2 t^2 + \cdots) \tag{4}$$

of (3), is

$$\alpha(\alpha - 1) + 1.2\alpha + .24(h - 1) = 0.$$

This has the conjugate complex roots

$$\alpha = -.1 \pm \sqrt{.01 - .24(h-1)}, \tag{5}$$

which corresponds to $b_h \propto t^{-.1} \propto b^{-1/4}$. We conclude

THEOREM 1. The amplitude of any perturbation is inversely proportional to the fourth root of the radius, as the radius of a collapsing spherical cavity shrinks to zero.

In particular, we have *stability during expansion*, and *instability during collapse.*

A consideration of the imaginary part of α shows that the *phase* of the oscillation, which is approximately the same as that of

$$t^{\pm i\sqrt{h-1/2}} \cong b^{\pm 1\ 25 i\sqrt{h-1}},$$

undergoes a half-oscillation when b changes by a factor of $\pi/1.25\sqrt{h-1} \cong 2.5/\sqrt{h-1}$.

Of course, when b_h/b exceeds 0.2 or so, linear perturbation theory is no longer applicable, and Theorem 1 ceases to be physically applicable. (The same limitation applies to Binnie's discussion.)

4. Physical generalization. The preceding analysis can be extended by a few remarks to other physical situations.

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[3] This seems to be the only sense considered in [1]; see [1, formula (14)].

[4] For regular singular points and their indicial equations, see L. R. Ford, *Differential equations*, p. 184 ff. The inadequacy of Binnie's approximations should now be evident.

Remark 1. Surface tension has a negligible effect asymptotically. Thus its asymptotic contribution to the available energy is negligible, so that it does not affect the formula $b \propto t^{0.4}$, $b^{\cdot} \propto b^{-1.5}$, $b^{\cdot\cdot} \propto b^{-4}$. Now, writing (2) in the alternative form

$$b_h^{\cdot\cdot} + 3Cb^{-2.5}b_h^{\cdot} + 1.5C(h-1)b^{-5}b_h = 0, \tag{3'}$$

we see that (i) surface tension does not affect the coefficient of $b_h^{\cdot}$, and (ii) since surface tension alone causes oscillations of a period proportional to $b^{1.5}$, the contribution of $0(b^{-3})$ to the coefficient of b_h is negligible.

Independently, a simple consideration of *reversibility* (the invariance of (2) under $b \to -b$, $t \to -t$) shows that positive stability in the expansion phase must be matched by instability during collapse.

Remark 2. Similar calculations hold for collapsing cylindrical cavities; the 3 in (2) must be changed to 2, and the $p_h(\cos\phi)$ in (1) to $\cos h\phi$. The kinetic energy varies according to a less simple law[5], namely $b^2(b^{\cdot})^2 \ln b$. Neglecting the logarithmic term for simplicity, we set $b^{\cdot} \propto b^{-1}$, or $b \propto t^{0.5}$. This gives

$$b^{\cdot\cdot} + \frac{1}{t}b_h^{\cdot} + \frac{(h-1)}{4t^2}b_h = 0. \tag{6}$$

The indicial equation is $\alpha^2 + (h-1)/4 = 0$. Hence, to this approximation, the amplitude of perturbations neither grows nor decreases. In this sense, we have *neutral stability* to a first approximation; in the sense of *relative* amplitude b_h/b, we have instability during collapse, as in the spherical case. Finally, we have a half-oscillation in b_h when b changes by a factor of $2\pi/\sqrt{h-1}$.

5. General theorem. More generally, one can analyse the stability of solutions of ordinary linear differential equations near regular singular points, by writing down the dominant terms. Thus consider

$$t^n\frac{d^nx}{dt^n} + a_1t^{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_nx = 0. \tag{7}$$

The following result is self-evident, in view of (4).

THEOREM 2. Stability occurs as $t \uparrow 0$ if the real parts of all roots of the indicial equation[4] of (7) are positive; instability occurs if any of them is negative.

Applying the usual Routh-Hurwitz criteria[6] for this to the indicial equation, we derive as corollaries the following special *tests for stability* of (7).

Case $n = 1$. One condition, $a_1 < 0$.

Case $n = 2$. Two conditions, $a_1 < 1$ and $a_2 > 0$.

Case $n = 3$. Three conditions: $a_1 < 3$, $a_3 < 0$, $(3-a_1)(2-a_1+a_2) > -a_3$.

Case $n = 4$. Four conditions: $a_1 < 6$, $a_4 > 0$, $a_2 > 3a_1 - 11$, and

$$(6-a_1)(a_2-3a_1+11)(6-2a_1+a_2-a_3) > (6-2a_1+a_2-a_3)^2 + (6-a_1)^2a_4.$$

Remark. Note that, even when $n = 2$, the tests on the *coefficients* of (7) are more subtle than the usual tests.

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[5]T. E. Sterne, *J. Appl. Phys.* 21, 73–74 (1950).

[6]See J. V. Uspensky, *Theory of equations*, New York, 1949, Appendix III; by Theorem 2, $-\alpha$ must replace α in the usual test.

COROLLARY. Suppose that

$$bb_\lambda^{\cdot\cdot} + Ab^{\cdot}b_\lambda^{\cdot} + Cb^{\cdot\cdot}b_\lambda = 0, \tag{8}$$

and that $b \propto |t|^\beta$ as $b \to 0$, $t \to 0$. Then:

(i) If $C\beta(\beta - 1) < 0$, almost all $b_\lambda(t)$ increase without limit as $t \uparrow 0$ (or $t \downarrow 0$).

(ii) If $C\beta(\beta - 1) > 0$ and $A\beta > 1$, then all $b_\lambda(t)$ increase without limit as $t \uparrow 0$ (or $t \downarrow 0$).

(iii) If $C\beta(\beta - 1) > 0$ and $A\beta < 1$, then all $b_\lambda(t)$ tend to zero as $t \uparrow 0$ (or $t \downarrow 0$).

Proof. Substituting $b = Kt^\beta$ in (8) and simplifying, we get

$$t^2 b_\lambda^{\cdot\cdot} + A\beta t b_\lambda^{\cdot} + C\beta(\beta - 1)b_\lambda = 0. \tag{9}$$

This has a regular singular point at $t = 0$, with the indicial equation

$$\alpha^2 + (A\beta - 1)\alpha + C\beta(\beta - 1) = 0. \tag{10}$$

The two roots α_1, α_2 of this are

$$\alpha_i = \frac{1 - A\beta}{2} \pm \frac{1}{2}\sqrt{(A\beta - 1)^2 - 4C\beta(\beta - 1)}\ .$$

In Case (i), α_1 and α_2 have opposite sign, and so the general solution $b_\lambda = c_1 t^{\alpha_1} + c_2 t^{\alpha_2}$ of (8) (logarithmic terms[4] do not alter this) increases without limit as $t \uparrow 0$. In Cases (ii)-(iii), α_1 and α_2 both have the same sign as $1 - A\beta$, whence the conclusions are still obvious.

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ELECTROMAGNETIC WAVE PROPAGATION IN BOUNDED ELECTRON BEAMS*

BY PHILIP PARZEN (*New York University*)

The linearized Maxwell's equations in *MKS* units for the field quantities in electron beams are:

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}, \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \sigma\mathbf{E} + j\omega\epsilon_0\mathbf{E}, \tag{2}$$

$$\nabla \cdot \mathbf{J} + j\omega\rho = 0, \tag{3}$$

$$j\omega\mathbf{V} + \mathbf{V}_0 \cdot \nabla\mathbf{V} + \mathbf{V} \cdot \nabla\mathbf{V}_0 = \frac{e}{m}[\mathbf{E} + \mathbf{V} \times \mathbf{H}_0 + \mathbf{V}_0 \times \mathbf{H}], \tag{4}$$

$$\mathbf{J} = \rho_0\mathbf{V} + \rho\mathbf{V}_0\ . \tag{5}$$

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