function of the limit stresses and a solution of problem (P). Because of the uniqueness Φ is given by (2.4).

This verification as well as the uniqueness proof are by no means superfluous. In fact, the solution ceases to be unique if the condition of boundedness is dropped. Consider for example the function

$$\Phi^*(z, r) = r^4(1 - z^2 - r^2)[(z - 1)^2 + r^2]^{-7/2}, \tag{5.1}$$

which satisfies (2.1) in H and assumes the boundary value 0 everywhere, except at B, where $\Phi^* = O(b^{-2})$. This stress function corresponds to a singular self-equilibrated system of forces at B. Superposing (2.4) and (4.2) we obtain a function which fulfills all conditions of problem (P) except boundedness. In the terminology of E. Sternberg and F. Rosenthal [5], who treated the case of concentrated loads, it is a "pseudosolution" of the physical problem.

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ON SUBHARMONIC SYNCHRONIZATION OF NEARLY-LINEAR SYSTEMS*

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When a linear system is driven by an external force, one has become accustomed to expect a frequency response related to the combination of the driving frequency and the natural frequency of the system. If one of the elements of the system is non-linear, but only slightly so, we may expect the frequency of the periodic response to behave like that of a linear system. For a physical system which is described by the non-linear differential equation

$$\ddot{y} + \epsilon f(y)\dot{y} + \omega_0^2 y = F(t), \tag{1}$$

where y is of the order unity, and ϵ is very much less than unity, this is true for all but a small interval in the range of driving frequencies. It has been shown in several special cases, notably for the van der Pol equation [1], that if this system is driven by a force $A \cos(\omega t + \varphi)$, the periodic response of the system for values of $\omega^2 - \omega_0^2$ small, say of order ϵ, is *not* a combination or beat frequency, but rather a function with frequency

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exactly that of the driving function, i.e. with circular frequency ω. This is the phenomenon called synchronization (or entrainment, or locking-in). Various experiments involving synchronization are described in the treatise of Minorsky [2].

A further investigation [3] of the particular case, the van der Pol equation in which $f(y) = -(1 - y^2)$, has shown that a stable synchronized subharmonic response will also occur when the system is driven by a frequency near to three times the natural frequency of the system ("natural frequency" of the non-linear system refers to the natural frequency of the system for $\epsilon = 0$). Thus, when the driving force is of the form $A \cos(3\omega t + \varphi)$, and when $\omega^2 - \omega_0^2$ is small, a periodic response of circular frequency ω will occur.

It would seem natural to ask, then, under what conditions synchronization will appear for systems described by (1) for other driving forces. Suppose the system is driven by a function of the form $A \cos(n\omega t + \varphi)$,—

$$\ddot{y} + \epsilon f(y)\dot{y} + \omega_0^2 y = A \cos(n\omega t + \varphi), \tag{2}$$

where A is of order unity and suppose, further, that $f(y)$ is such that periodic solutions analytic in ϵ are known to exist (the conditions necessary for such solutions are given by Friedrichs who has used a method of Poincaré [4]). Under what conditions will subharmonic synchronization as described above be found? In the following discussion n will be considered greater than unity.

The method to be used is a simple generalization of that given in [3]. The function y is expanded in a series in ϵ, the first term being the "linearized" solution.

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots \tag{3}$$

y_0 will then contain two constants and these are determined to order ϵ by the casting out of secular solutions. A condition for the existence of synchronization and its interval can then be determined.

Proceeding in this fashion, then, (2) is re-written in the form

$$\ddot{y} + \omega^2 y = A \cos(n\omega t + \varphi) + \epsilon f(y)\dot{y} + (\omega^2 - \omega_0^2)y.$$

Substituting the series (3), the equation of degree zero in ϵ is found to be

$$\ddot{y}_0 + \omega^2 y_0 = A \cos(n\omega t + \varphi) \tag{4}$$

and

$$y_0 = \alpha \cos \omega t + A_1 \cos(n\omega t + \varphi), \tag{5}$$

where $A/\omega^2(1 - n^2) = A_1$. The first degree in ϵ equation is

$$\ddot{y}_1 + \omega^2 y_1 = -f(y_0)\dot{y}_0 + cy_0, \tag{6}$$

where

$$c = (\omega^2 - \omega_0^2)/\epsilon.$$

According to the method of Poincaré, it is now necessary to determine coefficients of $\cos \omega t$ and $\sin \omega t$ on the right hand side of (6), and to equate these to zero. The function $f(y_0)$ is now periodic and may be expanded in a Fourier series, say

$$f(y_0) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} (a_m \cos m\omega t + b_m \sin m\omega t). \tag{7}$$

Then for the right hand side of (6) we have

$$\cos \omega t[c\alpha + \tfrac{1}{2}\omega\alpha b_2 + \tfrac{1}{2}\omega n A_1 \sin \varphi(a_{n-1} + a_{n+1}) + \tfrac{1}{2}\omega n A_1 \cos \varphi(b_{n+1} + b_{n-1})]$$
$$+ \sin \omega t[\tfrac{1}{2}\omega\alpha a_0 - \tfrac{1}{2}\omega\alpha a_2 + \tfrac{1}{2}\omega n A_1 \cos \varphi(a_{n-1} - a_{n+1}) - \tfrac{1}{2}\omega n A_1 \sin \varphi(b_{n-1} - b_{n+1})]$$

+ terms in higher harmonics.

Thus for c to be other than zero

$$\tfrac{1}{2}\omega\alpha b_2 + \tfrac{1}{2}\omega n A_1[\sin \varphi(a_{n-1} + a_{n+1}) + \cos \varphi(b_{n-1} + b_{n+1})]$$

must be other than zero. In order to obtain α and φ the coefficients of $\cos \omega t$ and $\sin \omega t$ are set equal to zero and solved simultaneously:

$$c\alpha + \tfrac{1}{2}\omega\alpha b_2 + \tfrac{1}{2}\omega n A_1[\sin \varphi(a_{n-1} + a_{n+1}) + \cos \varphi(b_{n-1} + b_{n+1})] = 0, \tag{8}$$

$$\tfrac{1}{2}\alpha a_0 - \tfrac{1}{2}\alpha a_2 + \tfrac{1}{2}n A_1[\cos \varphi(a_{n-1} - a_{n+1}) - \sin \varphi(b_{n-1} - b_{n+1})] = 0. \tag{9}$$

This appears somewhat involved but in at least one important case, the coefficients a_2, b_2, a_{n-1}, b_{n-1}, a_{n+1}, b_{n+1}, are such that these relationships become trivial. In the van der Pol equation, $f(y_0) = -(1 - y_0^2)$. Expanding this with $y_0 = \alpha \cos \omega t + A_1 \cos (n\,\omega t + \varphi)$, the Fourier coefficients are found to be (for $n > 3$)

$$\begin{aligned} &\tfrac{1}{2}a_0 = -(1 - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}A_1^2), \qquad a_2 = \tfrac{1}{2}\alpha^2, \qquad b_2 = 0, \\ &a_{n-1} = \alpha A_1 \cos \varphi, \qquad a_{n+1} = \alpha A_1 \cos \varphi, \\ &b_{n-1} = -\alpha A_1 \sin \varphi, \qquad b_{n-1} = -\alpha A_1 \sin \varphi. \end{aligned} \tag{10}$$

Inserting these quantities in (8) and (9) one obtains

$$c\alpha = 0, \qquad 4(1 - \tfrac{1}{2}A_1^2) = \alpha^2. \tag{11}$$

There is no range of synchronization, and "subharmonic resonance" (see [1] p. 323) occurs at $\omega = \omega_0$ only with amplitude α. In particular one observes that for $A = 0$, the known first approximation to the free vibration amplitude, $\alpha = 2$, is assumed.

For $n = 3$, the equations reduce to exactly those obtained in [3], but in this case, as was shown, there is a region of synchronization.

It should be made quite clear that we have chosen a rather restricted definition of subharmonic response. A driving force of circular frequency $n\omega$ might also produce a response of circular frequency say $n\,\omega/r$, where r is an integer less than n and this might also be termed subharmonic response. A further generalization can also be made in the damping term so that it takes on the form $\epsilon f(y, \dot{y})$. The system is still of the nearly linear type and the analysis is quite the same. The equation is re-written

$$\ddot{y} + \epsilon f(y, \dot{y}) + r^2\omega^2 y = A \cos (n\omega t + \varphi) + (r^2\omega^2 - \omega_0^2)y. \tag{12}$$

The series $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 \cdots$ is substituted in (12) and one obtains as conditions for existence of a synchronization interval

$$c_1 = d_r/\alpha_r, \qquad e_r = 0, \tag{13}$$

where $c_1 = r^2\omega^2 - \omega_0^2/\epsilon$, α_r = amplitude of response with circular frequency $r\omega$, and d_r and e_r are the Fourier coefficients of order r in the expansion of $f(y, \dot{y})$ as a function

of the zero degree solution, namely

$$y_0 = \alpha_r \cos r\omega t + A_1 \cos (n\omega t + \varphi), \qquad A_1 = A/\omega^2(r^2 - n^2).$$

It should be emphasized that the preceding work has been carried out on the assumption that the forcing term $F(t)$ in (1) is of order unity. Synchronization effects have been sought for, therefore, only in the terms of the solution of this order of magnitude.

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A NOTE ON THE PATTERSON FUNCTIONS*

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The following brief note was suggested by a paper of Hartman and Wintner [1], dealing with the Patterson function $P(x)$, defined by

$$P(x) = L^{-1} \int_0^L \rho(t)\rho(t + x)\, dt, \tag{1}$$

where $\rho(x)$ denotes a positive periodic function of period L. If $\rho(x)$ possesses a finite number N of (relative) maxima on the interval $0 \leqq x < L$, then, as a consequence of what the above authors refer to as the Patterson principle in crystallography (see [2]), the function $P(x)$ would have at most $N(N - 1)$ maxima on the interval $0 < x < L$. (Obviously, the function $P(x)$ is a positive even periodic function of period L, with absolute maxima occurring at the points $x = 0, \pm L, \pm 2L, \cdots$.) By constructing counter-examples for the case $N = 2$, the authors show that $P(x)$ can have more than the predicted two peaks on $0 < x < L$ and conclude that the Patterson principle cannot be valid as a general mathematical theorem.

The present note will deal with the case $N = 1$, so that $\rho(x)$ corresponds, in the terminology of Patterson ([2], p. 521), to an electron density which is itself an "atomic function" possessing a single peak on $0 \leqq x < L$. It will be shown that not only is the Patterson principle (which would here deny the existence of any peaks of $P(x)$ on $0 < x < L$; see *loc. cit.*, middle of page 522) false even in this case, but that $P(x)$ may, in fact, have *any* specified finite number of peaks on $0 < x < L$.

To this end, let n denote an arbitrary positive integer and divide the interval $0 \leqq x \leqq L$ into $2n$ parts of length $d = L/2n$. Next define the step-function $\rho(x)$

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