exponentially growing term as well as a damping term. With the exception of very special initial conditions, both terms will be present and the tube will be unstable.

4. Further remarks. It appears that the behavior of the higher order perturbation terms cannot be obtained as simply as those discussed above. These may require explicit determination of the function $F_0, F_1, F_2, \ldots$. It should be noted, however, that if $F_0, \ldots, F_{n-1}$ have been found, $D_n$ can be determined by quadratures. Furthermore, the differential equation for $F_n$ will be of the form

$$F_n^{IV} - D_n^2 F_n = \text{previously determined functions}.$$  

The Green's function for this equation, in the case of simply supported ends is known [5], and can be determined for the other boundary conditions by standard methods. Thus $F_n$ can be found by integration.

These comments indicate that the perturbation terms can be computed step-by-step by quadratures. Furthermore, in the supported end cases the critical velocity can be determined beforehand and perturbation from this point in powers of $D$ will serve as a check on the perturbation solution in terms of powers of $u$.

References


ON AN OSCILLATION CRITERION OF DE LA VALLÉE POISSON*

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An oscillation criterion of de la Vallée Poussin¹ on homogeneous, linear differential equations of order $n$, when particularized² to $n = 2$, runs as follows: Let both coefficient functions of

$$x'' + g(t)x' + f(t)x = 0$$  

be real-valued and continuous on a $t$-interval and suppose that (1) has a solution $x(t) \neq 0$ which vanishes for at least two points of that $t$-interval, say at $t = 0$ and at $t = h > 0$ (so that

$$x(0) = 0, \quad x(h) \neq 0,$$  

where, without loss of generality, $x(t) \neq 0$ when $0 < t < h$). Then

$$1 < M_1 h + M_2 h^2 / 2,$$  

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²F. Tricomi, Equazioni differenziali, 1948, p. 110.
where
\[ M_1 = \max_{0 \leq t \leq h} |g(t)|, \quad M_2 = \max_{0 \leq t \leq h} |f(t)|. \] (4)

It will be shown in this note that (3) can be improved to
\[ 1 < M_1 h/2 + M_2 h^2/6. \] (5)
Incidentally, it will also follow that the assumption that \( g(t) \) and \( f(t) \) are real-valued, an assumption used in de la Vallée Poussin's proof, can be omitted.

It will also follow that, instead of assuming (5) for the numbers (4), it is sufficient to assume
\[ 1 < \int_0^h |g(t)| \, dt + \frac{h}{4} \int_0^h |f(t)| \, dt \] (6)
for the coefficient functions \( g, f \) of (1).

In order that (1) has a solution \( x(t) \neq 0 \) satisfying (2), a necessary condition for (1), containing both (5) and (6), can be formulated as follows:
\[ h < \max \left\{ \int_0^h t \, |g(t)| \, dt, \int_0^h (h - t) \, |g(t)| \, dt \right\} + \int_0^h t(h - t) \, |f(t)| \, dt. \] (7)
If \( g(t) \equiv 0 \), then (1) reduces to
\[ x'' + f(t)x = 0 \] (8)
and (6) to
\[ 4 < h \int_0^h |f(t)| \, dt. \] (9)
Hence the criterion (7) to be proved generalizes the following well-known fact, which goes back to Liapounoff: If \( f(t) \) is continuous for \( 0 \leq t \leq h \), then a necessary condition for (8) to have a solution \( (x \neq 0) \) satisfying (2) is (9).

The proof of the statement concerning (7) depends on a device used by Nehari for a similar purpose and proceeds as follows:

First, if \( x(t) \), where \( 0 \leq t \leq h \), is any function possessing a continuous second derivative and satisfying (2), then it is readily verified that
\[ hx(t) = (h - t) \int_0^t s x''(s) \, ds + t \int_t^h (h - s) x''(s) \, ds \]
is an identity for \( 0 \leq t \leq h \). A differentiation gives
\[ hx'(t) = -\int_0^t s x''(s) \, ds + \int_t^h (h - s) x''(s) \, ds. \] (10)
Next, it is clear from (2) that both \( \mu t \) and \( \mu (h - t) \) are majorants for \( |x(t)| \) if \( 0 \leq t \leq h \) and
\[ \mu = \max_{0 \leq t \leq h} |x'(t)|. \] (11)

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3P. Hartman and A. Wintner, *On an oscillation criterion of Liapounoff*, Amer. J. Math. 73, 885–890 (1951), where further references will be found.

Thus
\[ |x(t)| \leq \mu \varphi(t), \tag{12} \]
where
\[ \varphi(t) = \min (t, h - t), \tag{13} \]
and it is clear that the \( \leq \) in (12) is a < for some \( t \).

If \( x'' \) is substituted from (1) into (10) (as \( -gx' - fx \)), it is seen from (11) and (12) that
\[
h |x'(t)| \leq \mu \int_0^t s(|g(s)| + \varphi(s) + f(s)) \, ds + \mu \int_t^h (h - s)(|g(s)| + \varphi(s) + f(s)) \, ds. \tag{14} \]

Let the maximum of \( |x'(t)| \) on \( 0 \leq t \leq h \) be attained at \( t = t_0 \). Then (14) and the remark made after (13) imply that
\[
h < \int_0^{t_0} s(|g(s)| + \varphi(s) + f(s)) \, ds + \int_{t_0}^h (h - s)(|g(s)| + \varphi(s) + f(s)) \, ds, \tag{15} \]
since \( |x'(t_0)| = \mu \), by (11), and since \( \mu \neq 0 \), by (2), where \( x(t) \neq 0 \).

The definition (13) of \( \varphi(t) \) shows that both \( s \varphi(s) \) and \( (h - s)\varphi(s) \) are majorized by \( s(h - s) \) for \( 0 \leq s \leq h \). Hence (15) can be written as
\[
h < \int_0^{t_0} s \, g(s) \, ds + \int_{t_0}^h (h - s) \, g(s) \, ds + \int_0^h s(h - s) \, f(s) \, ds. \tag{16} \]

If the sum \( S \) of the first two integrals is considered as a function of \( t_0 \), it is seen that its derivative is
\[
\frac{dS}{dt_0} = t_0 \, g(t_0) - (h - t_0) \, g(t_0) = (2t_0 - h) \, g(t_0),
\]
which is non-positive or non-negative according as \( t_0 \leq \frac{1}{2} h \) or \( t_0 \geq \frac{1}{2} h \). Hence the maximum of \( S = S(t_0) \) for \( 0 \leq t_0 \leq h \) is attained either at \( t_0 = 0 \) or at \( t_0 = h \). This fact, when combined with (16), leads to the criterion (7).

Ad (6). Since neither \( s \) nor \( h - s \) exceeds \( h \) for \( 0 \leq s \leq h \) and since \( s(h - s) \leq (h/2)^2 \), condition (6) is contained in (7).

Ad (5). In view of (4), the inequality (7) implies that
\[
h < M_1 \max \left\{ \int_0^h s \, ds, \int_0^h (h - s) \, ds \right\} + M_2 \int_0^h s(h - s) \, ds.
\]
Here the factor of \( M_1 \) is \( h^2/2 \) and that of \( M_2 \) is \( h^2/6 \). Hence (5) is contained in (7).