SOLUTIONS OF THE HYPER-BESSEL EQUATION*

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In problems of hydrodynamic stability involving axial symmetry, it is sometimes necessary to find the solutions of a differential equation of the type

\[ L_n^f = 0 \]

in which \( n \) is a positive integer, and (with \( D \equiv d/dr \))

\[ L_1 = D^2 + r^{-1}D - r^{-2} - \lambda^2 \]

is the Bessel operator of the first order. In this note, solutions of the equation \( L_n^f = 0 \) (1)

in which

\[ L_n = D^2 + r^{-1}D - p^2r^{-2} + k^2 \]

will be given explicitly. The theorem one seeks to establish is the following: If \( p \) (taken to be positive for convenience) is not an integer, the solutions of Eq. (1) are \( r^m J_{n+p+m}(kr) \) in which \( m = 0, 1, 2, \cdots, n - 1 \); otherwise they are \( r^m J_{n+p+m}(kr) \) and \( r^m N_{n+p+m}(kr) \), with \( m \) ranging over the same integers. The symbols \( J \) and \( N \) stand for the Bessel function and the Neumann function, respectively.

Proof: It is known that the solutions of \( L_n^f = 0 \) are \( J_{n+p}(kr) \) for \( p \) not equal to an integer and \( J_n(kr) \) and \( N_n(kr) \) for \( p \) equal to an integer. Thus it suffices to show that if \( r^sZ_{n+s}(kr) \) (in which \( Z \) stands for either \( J \) or \( N \)) satisfies \( L^s+1 = 0 \), then \( r^sZ_{n+s+1}(kr) \) satisfies \( L^s+1 f = 0 \), since the proof for \( r^mJ_{n+p+m}(kr) \) is identical with that for \( r^mN_{n+p+m}(kr) \). This will be accomplished if one can show that \( L_n r^s Z_{n+s+1}(kr) \) is equal to a constant times \( r^s Z_{n+s}(kr) \). By straightforward differentiation one has

\[ L_n r^{s+1}Z_{n+s+1}(kr) = r^{s+1}L_n Z_{n+s+1}(kr) + s(s + 1)r^{s-1}Z_{n+s+1}(kr) \]

\[ + 2(s + 1)r^s D Z_{n+s+1}(kr) + (s + 1) r^{s-1} Z_{n+s+1}(kr) = r^{s+1}L_n Z_{n+s+1}(kr) + (s + 1) r^{s-1} Z_{n+s+1}(kr) \]

\[ + 2(s + 1)r^s D Z_{n+s+1}(kr). \]

But

\[ L_n = L_{n+1} + \frac{2p(s + 1) + (s + 1)^2}{r^2} \]

and [1]

\[ D Z_{n+s+1}(kr) = k \left[ - \frac{s + 1}{kr} Z_{n+s+1}(kr) + Z_{n+s+1}(kr) \right]. \]

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So

\[ L_{p+1} Z_{p+1}(kr) = r^{p+1} L_{p+1} Z_{p+1}(kr) + [2p(s + 1) + (s + 1)^2] r^{p-1} L_{p+1} Z_{p+1}(kr) \]

\[ + (s + 1)^2 r^{p-1} Z_{p+1}(kr) + 2(s + 1) r^p \left[ - \frac{p + s + 1}{kr} Z_{p+1}(kr) + Z_{p+1}(kr) \right] \]

\[ = 2(s + 1) r^p Z_{p+1}(kr) \]

since

\[ L_{p+1} Z_{p+1}(kr) = 0 \]

by definition of Z.

Dr. Y. C. Fung of the California Institute of Technology communicated to the writer a different proof of the present result by means of Almansi's theorem [2] on hyperharmonic functions. His proof will not be presented here.

It may be noted that since [1]

\[ Z_{p-1}(kr) + Z_{p+1}(kr) = \frac{2p}{kr} Z_p(kr) \quad (2) \]

and since by the theorem just proved \( rZ_{p+1}(kr) \) and \( Z_p(kr) \) are solutions of

\[ L^2 f = 0, \quad (3) \]

it follows from Eq. (2) that \( rZ_{p-1}(kr) \) is also a solution of Eq. (3). In fact, by repeated use of Eq. (2) and a similar one obtained by changing \( p \) to \( -p \) in Eq. (2), it can be proved that if the \( m \) in the subscripts of the solutions given in the theorem is changed to \( -m \), the results will still be solutions of Eq. (1). These solutions are of course not independent of the ones given in the statement of the theorem.

REFERENCES

[2] E. Almansi, Sull' integrazione dell' equazione differenziale \( \Delta u = 0 \), Annali di Matimatica, (III) 2 (1899)