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ON THE CLASSIFICATION OF THE ORDINARY DIFFERENTIAL EQUATIONS OF FIELD THEORY*

BY

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Introduction. Despite the great amount of work that has been done on the solution of the partial differential equations of mathematical physics by separation of variables, the subject remains in an unfinished state. In a comprehensive treatment of the subject, the first problem is to determine separability conditions for the Laplace and Helmholtz equations and to find the coordinate systems that satisfy these conditions. The second problem is to tabulate all the separation equations. The first of these questions has been studied in recent papers [1], while the second is considered in the present contribution.

Attempts have been made in the past to subsume all the separation equations under a single general equation; but these formulations have not been satisfactory, since they include equations that are never encountered in physics and they exclude equations that are definitely needed. For instance, the hypergeometric equation includes both Bessel and Legendre equations but excludes most of the separation equations that occur in field theory. There seems to be a widespread idea that Bôcher's "generalized Lamé equation" with five singularities includes all the needed equations as confluent cases. According to Ince [2], "This systematization was suggested by the discovery of Klein and Bôcher that the chief linear differential equations which arise out of the problems of mathematical physics can be derived from a single equation with five distinct regular singular points in which the difference between the two exponents relative to each singular point is $\frac{1}{2}$".

Similar statements are made by Forsyth [3] and by Whittaker and Watson [4]. Actually, only a few of the required equations are obtainable from the Bôcher prototype with five singularities ($h = 5$); though by employing a family of prototypes with $h = 1$ to $h = 7$, one can obtain the necessary equations.

The present study extends the general work of Bôcher [5] by actually finding the separation equations for the eleven simply-separable systems of Eisenhart [6], the eleven symmetric cyclide systems, and eighteen cylindrical systems. All separation equations for these coordinate systems reduce to nineteen distinct equations of the Bôcher type.

Coordinate systems. Important partial differential equations of classical field theory are the Laplace equation, the Poisson equation, the diffusion equation, and the scalar wave equation. The solution of all these equations reduces essentially to the solution of a single partial differential equation, the *Helmholtz equation*:

$$\nabla^2\varphi + \kappa^2\varphi = 0. \tag{1}$$

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For any orthogonal coordinate system (u^1, u^2, u^3) in Euclidean 3-space,

$$\nabla^2 \varphi = \frac{1}{g^{1/2}} \sum_{i=1}^{3} \frac{\partial}{\partial u^i} \left(\frac{g^{1/2}}{g_{ii}} \frac{\partial \varphi}{\partial u^i} \right). \tag{2}$$

If the Helmholtz equation is *simply separable,*

$$\varphi = U^1(u^1) \cdot U^2(u^2) \cdot U^3(u^3), \tag{3}$$

and the three separation equations are

$$\frac{1}{f_i} \frac{d}{du^i} \left(f_i \frac{dU^i}{du^i} \right) + U^i \sum_{j=1}^{3} \alpha_j \Phi_{ij} = 0, \tag{4}$$

where α_j are the separation constants and Φ_{ij} are the elements of the Stäckel matrix [1].

Simple separability of the Helmholtz equation occurs in the confocal quadric systems of Eisenhart [6]. Apparently, these eleven coordinate systems exhaust the possibilities for the Helmholtz equation. If $\kappa^2 = 0$, however, we have the Laplace equation, which is R-separable [1] in a number of additional coordinates. The separation equations are again given by Eq. (4), but in place of Eq. (3) we have

$$\varphi = \frac{U^1(u^1) \cdot U^2(u^2) \cdot U^3(u^3)}{R(u^1, u^2, u^3)} . \tag{3a}$$

The only R-separable coordinates that give promise of practical applications are those having considerable symmetry. They include toroidal coordinates, bispherical coordinates, and the four systems based on elliptic functions [7]. We include the rotation cyclides of Bôcher [5] but deliberately omit his asymmetric cyclides as being too complex for use in physics and engineering.

In this paper, therefore, we shall consider the separation equations for the following coordinate systems:

I. Systems [8] *allowing simple separation of the Helmholtz and Laplace equations.*

Cylindrical systems.

1. Rectangular coordinates
2. Circular-cylinder coordinates
3. Elliptic-cylinder coordinates
4. Parabolic-cylinder coordinates

Rotational systems.

5. Spherical coordinates
6. Prolate spheroidal coordinates
7. Oblate spheroidal coordinates
8. Parabolic coordinates

Asymmetric systems.

9. Conical coordinates
10. Ellipsoidal coordinates
11. Paraboloidal coordinates

II. Systems allowing R-separation of the Laplace equation.

12. Tangent-sphere coordinates

13. Cardioid coordinates
14. Bispherical coordinates
15. Toroidal coordinates
16. Inverse prolate spheroidal coordinates
17. Inverse oblate spheroidal coordinates
18. 6-Sphere coordinates
19. Bi-cyclide coordinates (J1Rx)*
20. Flat-ring cyclide coordinates (J1Ry)
21. Disk-cyclide coordinates (J2R)
22. Cap-cyclide coordinates (J3R)

III. Cylindrical systems allowing simple separation of the Laplace equation in two dimensions only.

23. Tangent-cylinder coordinates (P1C)†
24. Cardioid-cylinder coordinates (P3C)
25. Hyperbolic-cylinder coordinates (P4C)
26. Rose coordinates (P5C)
27. Cassinian-oval coordinates (E2C)
28. Inverse Cassinian-oval coordinates (E3C)
29. Bi-cylindrical coordinates (E4C)
30. Maxwell-cylinder coordinates (E5C)
31. Logarithmic-cylinder coordinates (L1C)
32. Ln tan cylinder coordinates (L2C)
33. Ln cosh cylinder coordinates (L3C)
34. Inverse elliptic-cylinder coordinates (H2C)
35. *sn*-Cylinder coordinates (J1C)
36. *cn*-Cylinder coordinates (J2C)
37. Inverse *sn*-cylinder coordinates (J3C)
38. Ln *sn*-cylinder coordinates (J4C)
39. Ln *cn*-cylinder coordinates (J5C)
40. Zeta coordinates (J6C)

Bôcher equations. Except for one trivial case**, all the separation equations of field theory are second-order, linear differential equations that can be written in the form

$$\frac{d^2Z}{dz^2} + P(z)\frac{dZ}{dz} + Q(z) \qquad Z = 0, \tag{5}$$

where P and Q are rational functions of z. We now introduce the restriction [10] that

$$\begin{aligned} P(z) &= \frac{1}{2}\left[\frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \cdots + \frac{m_{n-1}}{z-a_{n-1}}\right], \\ Q(z) &= \frac{A_0 + A_1 z + \cdots + A_l z^l}{(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_{n-1})^{m_{n-1}}}, \end{aligned} \tag{6}$$

*See Reference [7].

†See Reference [9].

**The equation in time obtained by separation of the diffusion equation.

where m_i and l are positive integers. The equation has n singular points, which occur at $z = a_1, a_2, \cdots a_{n-1}, \infty$. The orders of the poles are $m_1, m_2, \cdots m_{n-1}$; and the order of the pole at ∞ is equal to the greater of two integers giving the orders of $[2z - z^2P(z)]$ and $z^4Q(z)$ for $z \to \infty$. The total order of the poles is $h = \sum_{i=1}^{n} m_i$.

Equations defined by (5) and (6) are called *Bôcher equations.* They are all obtainable from Bôcher prototypes; but it seems simpler to define them directly by Eq. (6) than to employ the rather round-about method used by Bôcher and Ince [2]. The direct approach also leads to a convenient method of classification [10] in terms of n instead of h.

A Bôcher equation may be specified by writing a sequence of integers representing the orders of the poles. The equation given by (5) and (6) is specified as $\{m_1, m_2, \cdots m_{n-1}, m_n\}$, where m_n is the order of the pole at ∞. The final integer m_n holds a privileged position: it must not be moved; though the other integers may be interchanged at will.

Some separation equations in their original form are not Bôcher equations, but they can be made into Bôcher equations by an elementary functional transformation of the independent variable. All equations obtained by such transformations may be regarded as *equivalent,* since if the solution of one is known, the solutions of the others are immediately evident. Among the equivalent equations obtained from a single equation there is in general only one Bôcher equation [10]. Thus the flexibility introduced by transformation of variables does not ordinarily cause any ambiguity in equation specification.

The only exception to this rule occurs with very simple equations having
(a) not more than two distinct singularities,
(b) $A_1 = A_2 = \cdots = A_l = 0$.

TABLE 1

Classification of Bôcher equations that are obtained by separation of variables in 40 coordinate systems

n	Name	Original	Degenerate Cases
4	Heine equation*	{1222}	
	Wangerin equation†	{1122}	{1121}
	Lamé wave equation	{1113}	{1112}, {1111}
3	Legendre wave equation	{123}	{122}, {121}
	Baer° wave equation	{114}	{113}
2	Bessel wave equation	{24}	{23}, {22}
	Weber equation	{14}	
	Elementary	{33}	{32}
1	Elementary	{04}	{01}
		Basic equations = 9,	
		Total equations = 19.	

*E. Heine, *Handbuch der Kugelfunctionen*, Berlin, 1878, p. 445.
†A. Wangerin, J. reine und angew. Math. vol. 82 (1877) p. 145.
°K. Baer, *Parabolische Koordinaten*, Frankfurt, 1888.

TABLE 2

The separation equations of field theory

For each designation, the first line gives the Bôcher equation, the second gives the transformation of variable that yields the equation of the third line. The fourth line presents the general solution.

n	Designation	Differential equations and solutions
4	$\{1222\}$	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{2}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z + A_2 z^2 + A_3 z^3}{(z-a_1)(z-a_2)^2(z-a_3)^2}\right]Z = 0,$ $z = sn^2\,\zeta,$ $\frac{d^2Z}{d\zeta^2} - \frac{sn\,\zeta(dn^2\,\zeta + k^2\,cn^2\,\zeta)}{cn\,\zeta\,dn\,\zeta}\frac{dZ}{d\zeta} + \left[2k^2\,sn^2\,\zeta - \alpha_2 - \alpha_3\frac{k'^4\,sn^2\,\zeta}{cn^2\,\zeta\,dn^2\,\zeta}\right]Z = 0,$ Heine functions.
4	$\{1122\}$	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z + A_2 z^2}{(z-a_1)(z-a_2)(z-a_3)^2}\right]Z = 0,$ $z = sn^2\,\zeta,$ $\frac{d^2Z}{d\zeta^2} + \frac{cn\,\zeta\,dn\,\zeta}{sn\,\zeta}\frac{dZ}{d\zeta} + \left[k^2\,sn^2\,\zeta - \alpha_2 - \alpha_3\left(k^2\,sn^2\,\zeta + \frac{1}{sn^2\,\zeta}\right)\right]Z = 0,$ Wangerin functions.
4	$\{1121\}$	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z}{(z-a_1)(z-a_2)(z-a_3)^2}\right]Z = 0,$ $z = sn^2\,\zeta,$ $\frac{d^2Z}{d\zeta^2} + \frac{cn\,\zeta\,dn\,\zeta}{sn\,\zeta}\frac{dZ}{d\zeta} - \left[\alpha_2 + \frac{1}{sn^2\,\zeta}\right]Z = 0,$ Wangerin functions.

TABLE 2—(*Continued*)

n	Designation	Differential equations and solutions
4	{1113}	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{1}{z-a_3}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z + A_2 z^2}{(z-a_1)(z-a_2)(z-a_3)}\right]Z = 0,$ $z = \zeta^2,$ $(\zeta^2 - b^2)(\zeta^2 - c^2)\frac{d^2Z}{d\zeta^2} + \zeta[2\zeta^2 - (b^2 + c^2)]\frac{dZ}{d\zeta} + [k^2\zeta^4 - p(p+1)\zeta^2 + (b^2 + c^2)q]Z = 0,$ Lamé wave functions.
4	{1112}	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{1}{z-a_3}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z}{(z-a_1)(z-a_2)(z-a_3)}\right]Z = 0,$ $z = \zeta^2,$ $(\zeta^2 - b^2)(\zeta^2 - c^2)\frac{d^2Z}{d\zeta^2} + \zeta[2\zeta^2 - (b^2 + c^2)]\frac{dZ}{d\zeta} + [(b^2 + c^2)q - p(p+1)\zeta^2]Z = 0,$ $Z = AE_p^q(\zeta) + BF_p^q(\zeta).$
4	{1111}	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{1}{z-a_3}\right]\frac{dZ}{dz} + \left[\frac{A_0}{(z-a_1)(z-a_2)(z-a_3)}\right]Z = 0,$ $z = \zeta^2,$ $(\zeta^2 - b^2)(\zeta^2 - c^2)\frac{d^2Z}{d\zeta^2} + \zeta[2\zeta^2 - (b^2 + c^2)]\frac{dZ}{d\zeta} + [(b^2 + c^2)q]Z = 0,$ $Z = AE_0^q(\zeta) + BF_0^q(\zeta).$

It can be shown [10] that all these simple cases that occur in field theory reduce to the equations {04} and {01}. We stipulate that if a set of Bôcher equations are equivalent, the specification of the set shall be that having smallest h and smallest n.

Separation equations. A study of the separation equations for the 40 coordinate systems listed in this paper shows that there are only 19 distinct separation equations. The 19 include all the degenerate cases obtained by allowing one or more separation

TABLE 2—(*Continued*)

n	Designation	Differential equations and solutions
3	$\{123\}$	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{2}{z-a_2}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z + A_2 z^2}{(z-a_1)(z-a_2)^2}\right]Z = 0,$ $z = \zeta^2,$ $(1-\zeta^2)\frac{d^2Z}{d\zeta^2} - 2\zeta\frac{dZ}{d\zeta} + \left[k^2c^2(1-\zeta^2) + p(p+1) - \frac{q^2}{1-\zeta^2}\right]Z = 0,$ Legendre wave functions.
3	$\{122\}$	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{2}{z-a_2}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z}{(z-a_1)(z-a_2)^2}\right]Z = 0,$ $z = \zeta^2,$ $(1-\zeta^2)\frac{d^2Z}{d\zeta^2} - 2\zeta\frac{dZ}{d\zeta} + \left[p(p+1) - \frac{q^2}{1-\zeta^2}\right]Z = 0,$ $Z = AP_p^q(\zeta) + BQ_p^q(\zeta).$
3	$\{121\}$	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{2}{z-a_2}\right]\frac{dZ}{dz} + \left[\frac{A_0}{(z-a_1)(z-a_2)^2}\right]Z = 0,$ $z = \zeta^2,$ $(1-\zeta^2)\frac{d^2Z}{d\zeta^2} - 2\zeta\frac{dZ}{d\zeta} - \frac{q^2}{(1-\zeta^2)}Z = 0,$ $Z = AP_0^q(\zeta) + BQ_0^q(\zeta).$

constants to be zero. They include also the equations obtained by separation of the *vector* Helmholtz equation [11] in the 5 coordinate systems in which this procedure has been shown to be possible. The distinct separation equations are designated in Table 1 and written in detail in Table 2. Note that by no means all of the equations have $h = 5$, as suggested by Ince.

Some of the equations of Table 2 are familiar and their solutions are well-known.

TABLE 2—(*Continued*)

n	Designation	Differential equations and solutions
3	{114}	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z - a_1} + \frac{1}{z - a_2}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z + A_2 z^2}{(z - a_1)(z - a_2)}\right]Z = 0,$ $z = \zeta,$ $\frac{d^2Z}{d\zeta^2} + \frac{1}{2}\frac{2\zeta - (b + c)}{(\zeta - b)(\zeta - c)}\frac{dZ}{d\zeta} + \frac{k^2\zeta^2 - p(p + 1) - (b + c)q}{(\zeta - b)(\zeta - c)}Z = 0,$ Baer wave functions.
3	{113}	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z - a_1} + \frac{1}{z - a_2}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z}{(z - a_1)(z - a_2)}\right]Z = 0,$ $z = \cos^2 \zeta,$ $\frac{d^2Z}{d\zeta^2} + (a - 2q \cos 2\zeta)Z = 0,$ Mathieu functions.
2	{24}	$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{2}{z - a_1}\right]\frac{dZ}{dz} + \left[\frac{A_0 + A_1 z + A_2 z^2}{(z - a_1)^2}\right]Z = 0,$ $z = \zeta^2,$ $\frac{d^2Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta} + \left(k^2\zeta + q^2 - \frac{p^2}{\zeta^2}\right)Z = 0,$ Bessel wave functions.

Others have been studied only slightly, if at all. In particular, separation of Laplace's equation ($\kappa = 0$) yields the Bessel equation, the Legendre equation, the Lamé equation. Separation of the Helmholtz equation ($\kappa \neq 0$) yields analogous but somewhat more complicated equations which we have called the Bessel wave equation, the Legendre wave equation, the Lamé wave equation. Despite their importance, very little attention has been given to these equations. There are also two distinct equations obtained from rotational cyclide coordinates, and these equations likewise need further study.

TABLE 2—(*Continued*)

n	Designation	Differential equations and solutions
2	{23}	$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{2}{z-a_1}\right]\frac{dZ}{dz}+\left[\frac{A_0+A_1z}{(z-a_1)^2}\right]Z=0,$ $z=\zeta^2,$ $\frac{d^2Z}{d\zeta^2}+\frac{1}{\zeta}\frac{dZ}{d\zeta}+\left(q^2-\frac{p^2}{\zeta^2}\right)Z=0,$ $Z=AJ_p(qz)+BY_p(qz).$
2	{22}	$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{2}{z-a_1}\right]\frac{dZ}{dz}+\left[\frac{A_0}{(z-a_1)^2}\right]Z=0,$ $z=\zeta^2,$ $\frac{d^2Z}{d\zeta^2}+\frac{1}{\zeta}\frac{dZ}{d\zeta}-\frac{p^2}{\zeta^2}Z=0,$ $Z=A\zeta^p+B\zeta^{-p}.$
2	{14}	$\frac{d^2Z}{dz^2}+\frac{1}{2(z-a_1)}\frac{dZ}{dz}+\left[\frac{A_0+A_1z}{z-a_1}\right]Z=0,$ $z=\zeta^2,$ $\frac{d^2Z}{d\zeta^2}+\left[q^2(p+\tfrac{1}{2})-\frac{q^4\zeta^2}{4}\right]Z=0,$ $Z=AD_p(q\zeta)+BD_{-p-1}(q\zeta).$
2	{33}	$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{3}{z-a_1}\right]\frac{dZ}{dz}+\left[\frac{A_1z+A_2z^2}{(z-a_1)^3}\right]Z=0,$ $z=\zeta^2,$ $\frac{d^2Z}{d\zeta^2}+\frac{2}{\zeta}\frac{dZ}{d\zeta}+p^2Z=0,$ $Z=\frac{1}{z}[A\sin p\zeta+B\cos p\zeta].$
2	{32}	$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{3}{z-a_1}\right]\frac{dZ}{dz}+\left[\frac{A_1z}{(z-a_1)^3}\right]Z=0,$ $z=\zeta^2,$ $\frac{d^2Z}{d\zeta^2}+\frac{2}{\zeta}\frac{dZ}{d\zeta}-\frac{p(p+1)}{\zeta^2}Z=0,$ $Z=A\zeta^p+B\zeta^{-(p+1)}.$

TABLE 2—(*Continued*)

n	Designation	Differential equations and solutions
1	{04}	$\dfrac{d^2Z}{dz^2} + A_0 Z = 0,$ $z = \zeta,$ $\dfrac{d^2Z}{d\zeta^2} + p^2 Z = 0,$ $Z = A \sin p\zeta + B \cos p\zeta.$
1	{01}	$\dfrac{d^2Z}{dz^2} = 0,$ $z = \zeta,$ $\dfrac{d^2Z}{d\zeta^2} = 0,$ $Z = A + B\zeta.$

Bibliography

1. P. Moon and D. E. Spencer, *Theorems on separability in Riemannian n-space*, Proc. Amer. Math. Soc. **3**, 635 (1952); *Recent investigations of the separation of Laplace's equation*, Proc. Amer. Math. Soc. **4**, 302 (1953)
2. E. L. Ince, *Ordinary differential equations*, Longmans, Green and Co., London, 1927, p. 494
3. A. R. Forsyth, *Theory of differential equations*, Cambridge University Press, 1902, Vol. IV, p. 161
4. E. T. Whittaker and G. N. Watson, *Modern analysis*, Cambridge University Press, 1946, p. 203
5. M. Bôcher, *Über die Reihenentwickelungen der Potentialtheorie*, B. G. Teubner, Leipzig, 1894
6. L. P. Eisenhart, *Separable systems of Stäckel*, Ann. of Math. **35**, 284 (1934)
7. P. Moon and D. E. Spencer, *Some coordinate systems associated with elliptic functions*, J. Franklin Inst. **255**, 531 (1953)
8. P. Moon and D. E. Spencer, *Separability conditions for the Laplace and Helmholtz equations*, J. Franklin Inst. **253**, 585 (1952); *Separability in a class of coordinate systems*, J. Franklin Inst., **254**, 227 (1952)
9. P. Moon and D. E. Spencer, *Cylindrical and rotational coordinate systems*, J. Franklin Inst. **252**, 327 (1951)
10. P. Moon and D. E. Spencer, *On the specification of Bôcher equations*, to be published in J. Franklin. Inst.
11. D. E. Spencer, *Separation of variables in electromagnetic theory*, J. Appl. Phys. 22, 386 (1951)