For the case of hydrostatic pressure,

$$N(t) = b^2 - t^2, \quad | t | < b; \quad N(t) = 0, \quad | t | > b, \quad T(t) = 0, \quad \text{all } t,$$

we can take

$$\Phi_3(\zeta) = b^2\Phi_1(\zeta) - \Phi_2(\zeta), \qquad \Psi_3(\zeta) = b^2\Psi_1(\zeta) - \Psi_2(\zeta),$$

or we can proceed as in the previous examples.

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ON THE INTEGRATION METHODS OF BERGMAN AND LE ROUX*

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Introduction. In a previous note [7] a correspondence was found between the representation of solutions, $u(x, y)$, of the linear hyperbolic differential equation

$$L(u) \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \tag{1}$$

in the forms

$$u(x, y) = 2 \int_0^1 E(x, y, t)f[\tfrac{1}{2}x(1 - t^2)] \frac{dt}{(1 - t^2)^{1/2}}, \tag{2}$$

and

$$u(x, y) = \int_0^x U(x, y, \alpha)g(\alpha)\, d\alpha. \tag{3}$$

The first representation is due to Bergman [3, 4, 6] and the second to Le Roux [1]. In Eq. (2), $E(x, y, t)$ is the even part of a solution for $-1 \leq t \leq 1$ of

$$(1 - t^2)(E_{yt} + aE_t) - \frac{1}{t}(E_y + aE) + 2xL(E) = 0, \tag{4}$$

such that, for $x \neq 0$,

$$\frac{(1 - t^2)^{1/2}(E_y + aE)}{xt} \tag{5}$$

is continuous for $t = 0$, and tends to zero for each (x, y) as t approaches $+1$; in Eq. (3), $U(x, y, \alpha)$ is a one-parameter family of solutions of Eq. (1) satisfying the characteristic condition

$$\frac{\partial U}{\partial y} + aU = 0 \quad \text{on} \quad x = \alpha. \tag{6}$$

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The correspondence in question is given by

$$\begin{cases} g(\alpha) = f(\tfrac{1}{2}\alpha), \\ U(x, y, \alpha) = \dfrac{E\left[x, y, \left(\dfrac{x-\alpha}{x}\right)^{1/2}\right]}{[\alpha(x-\alpha)]^{1/2}}. \end{cases} \tag{7}$$

However, it has the disadvantage that to functions $E(x, y, t)$, continuous at $t = 0$ and $t = 1$, there may correspond functions $U(x, y, \alpha)$ which are singular for $\alpha = x$ and $\alpha = 0$. The purpose of the present note is to give a second correspondence, whenever $f(0) = 0$, which avoids this difficulty. The modification necessary when $f(0) \neq 0$ will also be given, and an example discussed.

An alternative correspondence. An alternative to Eqs. (7) is suggested by considering the special case $a = b = c = 0$ in Eq. (1), the so-called wave equation when x and y are real. Clearly, see Eqs. (4) and (5), a possible choice for E is then $E(x, y, t) \equiv 1$, and a possible choice for U is $U(x, y, \alpha) \equiv 1$, see Eqs. (1) and (6). With these choices one obtains the same function $u(x, y)$ in Eqs. (2) and (3) provided f and g satisfy

$$2\int_0^1 f[\tfrac{1}{2}x(1-t^2)]\frac{dt}{(1-t^2)^{1/2}} = \int_0^x g(\alpha)\,d\alpha. \tag{8}$$

This last equation may be considered as an integral equation for f once g is given. To every[3] continuous g there corresponds a continuous f with $f(0) = 0$. For on setting $\lambda = x(1\ t^2)$ in the integral on the left one obtains

$$\int_0^x \frac{f(\frac{1}{2}\lambda)}{\lambda^{1/2}}\cdot\frac{d\lambda}{(x-\lambda)^{1/2}} = \int_0^x g(\alpha)\,d\alpha,$$

which is an Abel integral equation for $f(\frac{1}{2}\lambda)/\lambda^{\frac{1}{2}}$. Its continuous solution [2] is

$$f\left(\frac{1}{2}x\right) = \frac{x^{1/2}}{\pi}\int_0^x \frac{g(\beta)\,d\beta}{(x-\beta)^{1/2}}, \tag{9}$$

and clearly $f(0) = 0$.

On the other hand Eq. (8) is an integral equation for g once f is given. To every[4] continuously differentiable f with $f(0) = 0$ there corresponds a continuous g. For clearly if such a g exists it is given by

$$g(x) = \int_0^1 f'[\tfrac{1}{2}x(1-t^2)](1-t^2)^{1/2}\,dt, \tag{10}$$

and it is easily verified by integration that when $f(0) = 0$ this g satisfies Eq. (8).

This correspondence between f and g, expressed by Eqs. (9) and (10), is not one-to-one, since for example there are continuous functions g for which f is not continuously

[3]In Bergman's work f is required to be at least continuously differentiable, while Le Roux needs g to be just continuous.

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differentiable[5]. However, there is in fact such a correspondence between continuous functions g and continuous functions f, with $f(0) = 0$, for which the integral on the left side of Eq. (8) possesses a continuous x-derivative. When this correspondence is used, Eq. (10) is replaced by

$$g(x) = 2\frac{d}{dx}\int_0^1 f[\tfrac{1}{2}x(1 - t^2)]\frac{dt}{(1 - t^2)^{1/2}},$$

and corresponding (trivial) changes must be made in the sequel. Also there is a one-to-one correspondence between analytic functions f and g with $f(0) = 0$.

It will now be shown that in the general case of Eq. (1) a correspondence between E and U may be chosen so as to preserve the same correspondence between f and g. If for a given continuously differentiable f with $f(0) = 0$, a continuous function g is determined by means of Eq. (10), then f may be expressed in terms of g by Eq. (9). Inserting this expression for f into Eq. (2), after the change of integration variable $\lambda = x(1 - t^2)$ has been made in the latter, one has

$$u(x, y) = \frac{1}{\pi}\int_0^x E\left(x, y, \left(\frac{x - \lambda}{x}\right)^{1/2}\right)\left[\int_0^\lambda \frac{g(\beta)\, d\beta}{(\lambda - \beta)^{1/2}}\right]\frac{d\lambda}{(x - \lambda)^{1/2}}.$$

The integral on the right may be transformed by means of Dirichlet's inversion formula into

$$\frac{1}{\pi}\int_0^x g(\beta)\left[\int_\beta^x E\left(x, y, \left(\frac{x - \lambda}{x}\right)^{1/2}\right)\frac{d\lambda}{(x - \lambda)^{1/2}(\lambda - \beta)^{1/2}}\right] d\beta,$$

which has the form of the right member of Eq. (3) with

$$U(x, y, \beta) = \frac{1}{\pi}\int_\beta^x E\left(x, y, \left(\frac{x - \lambda}{x}\right)^{1/2}\right)\frac{d\lambda}{(x - \lambda)^{1/2}(\lambda - \beta)^{1/2}}, \tag{11a}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} E\left(x, y, \left(\frac{x - \beta}{x}\right)^{1/2}\cos\theta\right) d\theta, \tag{11b}$$

where $\lambda = x\sin^2\theta + \beta\cos^2\theta$.

It is easily shown[6] that if $E(x, y, t)$ satisfies Eq. (4) for $t \neq 0$, then for each β the function $U(x, y, \beta)$, defined by Eq. (11), satisfies Eq. (1) when $x \neq 0, \beta$. In addition, if $E(x, y, t)$ satisfies the condition (5) (which implies $E_y + aE \to 0$ as $t \to 0$) then $U(x, y, \beta)$ satisfies (6) on $x = \beta$. Unlike the previous correspondence (7), the present one has the property that to functions $E(x, y, t)$ which are continuous at $t = 0$ and $t = 1$, there correspond functions $U(x, y, \beta)$ which are continuous for $\beta = x$ and $\beta = 0$.

In order to invert Eq. (11a) so as to obtain E in terms of U this equation may be considered, for fixed x and y, as an Abel integral equation for E with variable β. Thus

$$E\left(x, y, \left(\frac{x - \gamma}{x}\right)^{1/2}\right) = U(x, y, x) - (x - \gamma)^{1/2}\int_\gamma^x \frac{U_\beta(x, y, \beta)}{(\beta - \gamma)^{1/2}}\, d\beta.$$

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[5]To $g(x) = 0$ for $x < x_0$ and $g(x) = (x - x_0)^\alpha$ for $x \geq x_0$, where $\alpha > 0$, there corresponds $f(x) = 0$ for $x < x_0$ and $f(x/2) = Ax^{1/2}.\ (x - x_0)^{\alpha + 1/2}$ where A is a non-zero constant. If $\alpha < \frac{1}{2}$, then f is not differentiable at $x = x_0$.

[6]If in particular E is analytic and even in t, then U is analytic for $x \neq 0$.

On setting $\gamma = x(1 - t^2)$ and changing the integration variable from β to ξ, where $\beta = x(1 - \xi t^2)$, one finds

$$E(x, y, t) = U(x, y, x) - xt^2 \int_0^1 \frac{U_\beta[x, y, x(1 - \xi t^2)]}{(1 - \xi)^{1/2}} \, d\xi. \tag{12}$$

Extension of the correspondence. So far only functions f have been considered for which $f(0) = 0$. When $f(0) \neq 0$ an extra term arises in Eq. (3) if the transformation of the previous section is applied to Eq. (2). Thus the latter may be written

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where

$$u_1(x, y) = 2f(0) \int_0^1 E(x, y, t) \frac{dt}{(1 - t^2)^{1/2}},$$

$$u_2(x, y) = 2 \int_0^1 E(x, y, t)F[\tfrac{1}{2}x(1 - t^2)] \frac{dt}{(1 - t^2)^{1/2}},$$

and $F[\frac{1}{2}x(1 - t^2)] = f[\frac{1}{2}x(1 - t^2)] - f(0)$. The preceding analysis may be applied to $u_2(x, y)$, since $F(0) = 0$. Thus $u_2(x, y)$ may be written in the form of Eq. (3) where $U(x, y, \alpha)$ is given by Eqs. (11) and, according to Eq. (10),

$$\begin{aligned} g(x) &= \int_0^1 F'[\tfrac{1}{2}x(1 - t^2)](1 - t^2)^{1/2} \, dt, \\ &= \int_0^1 f'[\tfrac{1}{2}x(1 - t^2)](1 - t^2)^{1/2} \, dt. \end{aligned} \tag{13}$$

The remaining term $u_1(x, y)$ may be written

$$\begin{aligned} u_1(x, y) &= 2f(0) \int_0^{\pi/2} E(x, y, \cos\theta) \, d\theta, \\ &= \pi f(0) U(x, y, 0), \end{aligned}$$

in view of Eq. (11b). Hence finally

$$u(x, y) = \pi f(0) U(x, y, 0) + \int_0^x U(x, y, \alpha) g(\alpha) \, d\alpha. \tag{14}$$

The first term on the right hand side of this equation is clearly a solution of Eq. (1) since $U(x, y, \alpha)$ is for all α.

An alternative form of Eq. (14) is obtained on writing

$$w(x) = 2 \int_0^1 f[\tfrac{1}{2}x(1 - t^2)] \frac{dt}{(1 - t^2)^{1/2}}, \tag{15}$$

so that $w(0) = \pi f(0)$ and $g(\alpha) = w'(\alpha)$, see Eq. (8). For then on integration by parts Eq. (14) becomes

$$\begin{aligned} u(x, y) &= \pi f(0) U(x, y, 0) + [U(x, y, \alpha) w(\alpha)]_{\alpha=0}^{\alpha=x} - \int_0^x U_\alpha(x, y, \alpha) w(\alpha) \, d\alpha, \\ &= U(x, y, x) w(x) - \int_0^x U_\alpha(x, y, \alpha) w(\alpha) \, d\alpha. \end{aligned} \tag{16}$$

Example. Eq. (16) is the form of solution obtained by von Mises and Schiffer [5, especially p. 258] in their discussion of Bergman's integration method as applied to the equation satisfied by the "modified" stream function of a compressible flow in the hodograph plane. This equation is a special case of Eq. (1) with[7] $a = b = 0, c = f(x + y)$. For the $U(x, y, \alpha)$ of the present paper they find

$$U(x, y, \alpha) = 1 + \sum_{n=1}^{\infty} G_n(x + y) \frac{(-1)^n(x - \alpha)^n}{2^n n!}, \tag{17}$$

where

$$G_0 = 1, \qquad G'_{n+1} = G''_n + fG_n, \tag{18}$$

and the prime denotes differentiation with respect to the argument $x + y$.

The same results are obtained with Bergman's representation, Eq. (2), with [4, especially p. 24]

$$E(x, y, t) = 1 + \sum_{n=1}^{\infty} t^{2n}x^n Q_n(x + y), \tag{19}$$

when w and f are related by Eq. (15). Here Q_n satisfies

$$Q_0 = 1, \qquad (2n + 1)Q'_{n+1} + Q''_n + fQ_n = 0. \tag{20}$$

It is easily verified that the functions $U(x, y, \alpha)$ and $E(x, y, t)$ given by Eqs. (17) and (19) are in fact connected by the general formulas (11b) and (12). For example, with $E(x, y, t)$ as in Eq. (19)

$$\begin{aligned}\int_0^{\pi/2} E\left(x, y, \left(\frac{x - \beta}{x}\right)^{1/2} \cos\theta\right) d\theta &= \frac{\pi}{2} + \sum_{n=1}^{\infty} (x - \beta)^n Q_n \int_0^{\pi/2} \cos^{2n}\theta \, d\theta, \\ &= \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(2n - 1)(2n - 3) \cdots 1}{2^n n!} (x - \beta)^n Q_n, \\ &= \frac{\pi}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x - \beta)^n}{2^n n!} G_n\right],\end{aligned}$$

and $G_n = (-1)^n (2n - 1)(2n - 3) \cdots 1 \cdot Q_n$ clearly satisfies Eqs. (18) when Q_n satisfies Eqs. (20).

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[7] Explicitly $x = \frac{1}{2}(\lambda + i\theta)$, $y = \frac{1}{2}(\lambda - i\theta)$.