ON THE STEADY-STATE THERMOELASTIC PROBLEM FOR THE HALF-SPACE*

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Summary. This paper deals with the determination of the steady-state thermal stresses and displacements in a semi-infinite elastic medium which is bounded by a plane. The problem is treated within the classical theory of elasticity and is approached by the method of Green. It is shown that the stress field induced by an arbitrary distribution of surface temperatures is plane and parallel to the boundary. If the surface temperature is prescribed arbitrarily over a bounded "region of exposure" and is otherwise constant, the problem reduces to the determination of Boussinesq's three-dimensional logarithmic potential for a disk in the shape of the region of exposure, whose mass density is equal to the given temperature. Moreover, it is found that there exists a useful connection between the solutions to Boussinesq's and to the present problem for the half-space. An exact closed solution, in terms of complete and incomplete elliptic integrals of the first and second kind, is given for a circular region of exposure at uniform temperature. Exact solutions in terms of elementary functions are presented for a hemispherical distribution of temperature over a circular region, as well as for a rectangle at constant temperature.

Introduction. Recent years have seen a revival of interest in the thermoelastic problem which has received repeated previous attention in the theory of elasticity.[1] Nevertheless, the existing literature on this subject is largely confined to two-dimensional problems and to problems characterized by polar or rotational symmetry.

A significant advance in connection with the general three-dimensional thermoelastic problem was made by Goodier [4], who considered a medium occupying the entire space; he reduced the computation of the thermal stresses due to an arbitrarily prescribed temperature distribution to the determination of the Newtonian potential for a mass distribution whose density coincides with the given temperature field. For domains other than the entire space, Goodier's approach merely supplies a particular solution of the thermoelastic equations and still necessitates the solution of an ordinary boundary-value problem in the theory of elasticity. A particular integral of the same form was employed earlier by Borchardt [5] in dealing with the special problem of the sphere.

Goodier's method was extended by Mindlin and Cheng [6] to the problem of the half-space with a traction-free boundary, subjected to an arbitrary temperature distribution in its interior. As an application of the extended scheme, the problem of the half-space with a uniformly heated (or cooled) spherical core is solved in [6].

In the present paper we return to the problem of the half-space but limit our attention

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[1]See, for example, [1], [2], [3] for extensive bibliographies. Numbers in brackets refer to the bibliography at the end of this paper.

to the case in which merely the surface temperature is prescribed arbitrarily while the temperature distribution in the interior conforms to the steady-state heat-flow equation. The temperature field is, therefore, harmonic throughout and is obtained as the solution of a Dirichlet problem for the domain under consideration.

Instead of applying the more general method of Mindlin and Cheng [6], we adopt an alternative approach which is especially suited to the restricted class of problems at hand. Thus, we establish first the particular (singular) solution of the thermoelastic field equations appropriate to a surface point-source of temperature situated at a point of the traction-free plane boundary. Following the method of Green in potential theory, and in analogy to Boussinesq's [10] treatment of the ordinary problem of the half-space, we then determine the solution corresponding to an arbitrary distribution of surface temperatures through an integration over the boundary. We find that the resulting stress field is plane and parallel to the boundary regardless of the particular distribution of surface temperatures. If the boundary values of the temperature are constant (zero) except for a bounded "region of exposure", the foregoing process, in conjunction with the Boussinesq-Papkovich stress functions, yields a reduction of the problem to the determination of the three-dimensional logarithmic potential for a disk in the shape of the region of exposure, whose mass density coincides with the given temperature.

The method of attack adopted here reveals an intimate connection between the present problem and Boussinesq's problem of the half-space subjected to a system of normal surface tractions which is identical with the given distribution of surface temperatures. In particular, the values of the displacements at the boundary in these two problems turn out to be proportional to each other. This observation may be of interest in connection with experimental stress analysis. Moreover, precisely the same potential arises in both problems; its required derivatives were determined by Love [7] in closed form for a variety of physically important cases. This enables us to reach directly closed exact solutions for a circular region of exposure under a uniform or hemispherical temperature distribution, as well as for a rectangular region at constant temperature.

Reference should also be made to a recent paper by Sadowsky [8] who considered a different thermoelastic problem associated with the half-space. In [8] a circular subregion of the boundary is assumed to be exposed to a heat input of uniform intensity. The interior of the body and the remainder of the boundary are supposed to be at zero temperature, the temperature of the region of exposure being infinite. While this problem would be amenable to an analogous mathematical treatment,[2] it bears no physical relation to the problem under present consideration.

Before turning to our main concern, we briefly recall the basic equations governing the thermoelastic problem and discuss certain general methods of integration.

The basic thermoelastic equations. Stress functions, thermoelastic potential. Consider an elastic medium occupying a regular region of space[3] D with the boundary B. Let $T(P)$ be the temperature field to which the body is subjected, P being a point of D. Without loss in generality we assume the body forces and the surface tractions to vanish identically in D and on B, respectively. The stresses and displacements induced by the

[2]Indeed, such a treatment would yield at once closed solutions also for a circular region subjected to a hemispherical distribution of the heat input, as well as for a uniform heat input over a rectangular region of exposure.

[3]See [9] for the definition of a "regular" region of space employed here. Note that the region need not be bounded or simply connected.

given temperature distribution, within the classical theory of elasticity, are characterized by the homogeneous stress equations of equilibrium, the linear isotropic stress-strain relations (including the temperature terms), and the linearized strain-displacement relations.[4] In indicial notation, and with reference to rectangular cartesian coordinates x_i, we have throughout D,

$$\tau_{ij,j} = 0, \tag{1}$$

$$\tau_{ij} = 2\mu\left[e_{ij} + \left(\frac{\nu}{1-2\nu}e_{kk} - \frac{1+\nu}{1-2\nu}\alpha T\right)\delta_{ij}\right], \tag{2}$$

$$2e_{ij} = u_{i,j} + u_{j,i}\,. \tag{3}$$

Here τ_{ij}, e_{ij}, and u_i are the cartesian components of stress, strain, and displacement, respectively, and the usual conventions for summation and space-differentiation have been used. The Kronecker delta is denoted by δ_{ij}, while μ, ν, and α designate the shear modulus, Poisson's ratio, and the coefficient of thermal expansion.

To (1), (2), (3) we must adjoin the boundary conditions which, in the absence of surface tractions, take the form,

$$\tau_{ij}n_j = 0 \quad \text{on} \quad B, \tag{4}$$

where n_i are the components of the unit outer normal of B. Furthermore, in order to assure the uniqueness of the solution (within an arbitrary additive rigid displacement field), the stresses and displacements are subject to certain regularity requirements which need not be listed here.[5]

Substitution of (3) into (2) and of (2) into (1), leads to the displacement equations of equilibrium which, in vector notation, appear as

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla\nabla\cdot\mathbf{u} = \frac{2(1+\nu)\alpha}{1-2\nu}\nabla T, \tag{5}$$

where $\mathbf{u}$ is the displacement vector,[6] ∇ and ∇^2 are the gradient and Laplacian operators, whereas the dot indicates scalar multiplication. The solution of the thermoelastic equilibrium problem thus, alternatively, requires the determination of a displacement field satisfying (5) and such that the associated field of stress—associated in the sense of (2) and (3)—meets the boundary conditions (4).

The general solution of (5) admits the representation

$$2\mu\mathbf{u} = \nabla(\phi + \mathbf{r}\cdot\boldsymbol{\psi}) - 4(1-\nu)\boldsymbol{\psi}, \tag{6}$$

where

$$\nabla^2\phi = \frac{2\mu\alpha(1+\nu)}{1-\nu}T, \qquad \nabla^2\boldsymbol{\psi} = 0, \tag{7}$$

and $\mathbf{r}$ is the position vector with components x_i. For $T(P) \equiv 0$, Equations (6), (7) reduce to the Boussinesq-Papkovich solution of the homogeneous displacement equations

[4]See, for example, [1], [2], [3] for general discussions of the thermoelastic problem.
[5]Cf. [9].
[6]Bold-face letters denote vectors.

of equilibrium in terms of four harmonic scalar stress functions.[7] The stress field generated by ϕ and $\boldsymbol{\psi}$ is readily obtained from (6) by means of (3), (2).

In the special case of rotational symmetry, say with respect to the x_3-axis, $T = T(\rho, z)$ and

$$\phi = \phi(\rho, z), \qquad \psi_1 = \psi_2 = 0, \qquad \psi_3 = \psi(\rho, z), \tag{8}$$

in which

$$\rho = (x_1^2 + x_2^2)^{1/2}, \qquad z = x_3 \tag{9}$$

are circular cylindrical coordinates. Here (6), (7) assume the form,

$$2\mu\mathbf{u} = \nabla(\phi + z\psi) - 4(1 - \nu)\mathbf{k}\psi, \tag{10}$$

$$\nabla^2\phi = \frac{2\mu\alpha(1 + \nu)}{1 - \nu} T, \qquad \nabla^2\psi = 0, \tag{11}$$

if $\mathbf{k}$ is a unit vector in the z-direction.

A particular solution of the Poisson equation for ϕ in (7) is given by the Newtonian potential[8]

$$\phi = V(P) = -\frac{\mu\alpha(1 + \nu)}{2\pi(1 - \nu)} \int_D \frac{T(Q)}{R(Q, P)} \, d\tau, \tag{12}$$

where $R(Q, P)$ is the distance between two points P and Q of D. On setting $\phi = V(P)$, $\boldsymbol{\psi} = 0$ in (6), we arrive at

$$2\mu\mathbf{u} = \nabla V, \tag{13}$$

which is the displacement field of Goodier's [4] particular solution of the thermoelastic field equations, here designated[9] by $[S_1]$. Solution $[S_1]$ is identified as appropriate to a distribution over D of centers of dilatation, whose density is proportional to the temperature $T(P)$. The displacement field of $[S_1]$ at the same time coincides with a Newtonian field of force generated by a mass distribution over D, the mass density of which is proportional to $T(P)$.

If D is the entire space, $[S_1]$ constitutes the complete solution $[S]$ to the thermoelastic equilibrium problem formulated earlier. Otherwise, the complete solution $[S]$ may be represented by

$$[S] = [S_1] + [S_2], \tag{14}$$

in which $[S_2]$ is the solution of a "residual problem" whose characterization is implicit in (14): $[S_2]$ in D must satisfy the field equations (1), (2), (3) for $T(P) = 0$ and must annul the surface tractions to which $[S_1]$ gives rise on B.

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[7]See [10], [11]. This solution was independently discovered by Neuber [12]; its completeness was established by Mindlin [13].

[8]In the event that D is not bounded, the behavior of $T(P)$ at infinity must be such as to assure the convergence of the integral in (12) and of its required space-derivatives.

[9]Throughout this paper, capital letters in brackets denote the displacement vector-field and the stress tensor-field of a solution to the field equations of elasticity theory. Equality, addition, multiplication by a scalar, differentiation and integration, are to be interpreted accordingly.

If D is the half-space, $[S_2]$ is derivable by differentiations from a potential which is the reflection of $V(P)$ in the plane boundary, as was shown by Mindlin and Cheng [6]. Here again the complete solution requires merely the determination of the Newtonian potential (12) for the given temperature distribution.

The steady-state problem. Solution by Green's method. We now return to a general domain (other than the entire space), but assume that the temperature is prescribed on the boundary only and conforms to the steady-state heat-flow equation in the interior of the body. Thus, let

$$\left.\begin{aligned} T &= f(Q) \quad \text{on} \quad B, \\ \nabla^2 T &= 0 \qquad \text{in} \quad D. \end{aligned}\right\} \tag{15}$$

If $G(Q, P)$ is Green's function of the first kind for the region D, the solution of the Dirichlet problem characterized by (15) admits the integral representation,[10]

$$T(P) = \int_B f(Q) T_0(Q, P)\, d\sigma, \tag{16}$$

where Q is a point on B, P is in D, and[11]

$$T_0(Q, P) = -\frac{1}{4\pi}\frac{\partial}{\partial n} G(Q, P) \equiv -\frac{1}{4\pi} \mathbf{n} \cdot \nabla_Q G(Q, P), \tag{17}$$

$\mathbf{n}$ being the outer unit normal of B. We may interpret $T_0(Q, P)$ as the steady-state temperature at P due to a surface point-source of temperature at Q, of unit strength.

The representation (16) of the solution to the temperature problem may be utilized to reach an analogous representation of the solution to the steady-state thermoelastic problem. Let $[S_0(Q, P)]$ be defined as follows:

(a) *For Q fixed on B, $[S_0(Q, P)]$ satisfies the field equations* (1), (2), (3) *with* $T = T_0(Q, P)$ *and is regular*[12] *in $D + B$ except for a point singularity at Q;*

(b) $[S_0(Q, P)]$ *meets the boundary conditions* (4);

(c) *the resultant force of the tractions acting on any surface enclosing Q, and lying wholly in D, vanishes;*

(d) $\tau_{ij}(Q, P) = O(r_Q^{-2})$ *as* $r_Q \to 0$, *where r_Q is the distance of P from Q.*

Conditions (a), (b), (c), (d) uniquely characterize $[S_0(Q, P)]$, as is clear from the proof of a generalized uniqueness theorem given in [9].[13] $[S_0(Q, P)]$ may be interpreted

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[10]See, for example, [14], p. 236 *et seq.*

[11]The letter Q under the del-operator signifies that the coordinates of P are to be held fixed in the formation of the gradient.

[12]Sufficient regularity conditions are that $[S_0(Q, P)]$ be continuous and continuously differentiable with respect to P in every closed regular subregion of $D + B$ not containing Q; if D is not bounded, the displacements and stresses are to vanish at infinity as r^{-1} and r^{-2}, respectively, where r is the distance from the origin.

[13]The difference between two solutions possessing these four properties evidently meets conditions (A), (B), (C), (D) of [9], p. 163, and hence must vanish identically. Requirement (c) precludes the presence of a concentrated load applied at the origin.

as the displacement vector and the local stress-state at P induced by a surface point-source of temperature at Q, of unit strength.

The complete solution of the steady-state thermoelastic problem may now be written in the form,

$$[S(P)] = \int_B f(Q)[S_0(Q, P)]\, d\sigma. \tag{18}$$

That $[S(P)]$, as defined here, indeed conforms to the formulation of the problem given previously, is intuitively plausible and readily confirmed analytically, provided the surface temperature $f(Q)$ is continuous and continuously differentiable on B. If $f(Q)$ is merely piecewise continuous on B, (18) supplies a natural and unique definition of the solution to the problem. Its solution in either case is reduced to the determination of the singular solution $[S_0(Q, P)]$ for the region under consideration.

If B extends to infinity (in which case $D + B$ is no longer a regular region of space),[14] the representation (18) remains valid provided B is a single analytic surface, e.g., $D + B$ is a half-space. In this instance, however, the behavior of $f(Q)$ at infinity must be such as to render the definite integral in (18) suitably convergent.

The steady-state problem for the half-space. At this stage we apply the general considerations of the preceding section to the steady-state thermoelastic problem for a semi-infinite body bounded by a plane. Here we find it expedient to depart from indicial notation. Thus, let (x, y, z) be rectangular cartesian coordinates, let D be the region $z > 0$, and B be the plane $z = 0$. Suppose that Σ is a bounded, regular subregion of B and let[15]

$$\left.\begin{aligned} T &= f(Q) \quad \text{for} \quad Q \quad \text{on} \quad B, \\ f(Q) &= 0 \quad \text{for} \quad Q \quad \text{not in} \quad \Sigma, \end{aligned}\right\} \tag{19}$$

where $f(Q)$ is assumed to be piecewise continuous in Σ. We shall refer to Σ as the "region of exposure".

For the half-space Green's function $G(Q, P)$ is known explicitly[16] and (17) here yields,

$$T_0(Q, P) = \frac{1}{2\pi}\frac{z}{R^3} \equiv -\frac{1}{2\pi}\frac{\partial}{\partial z}R^{-1}, \tag{20}$$

if R is the distance from a point $P(x, y, z)$ in D to a point $Q(\xi, \eta, 0)$ on B. The solution for the steady-state temperature distribution, in the integral representation (16), presently becomes,

$$T(P) = \frac{z}{2\pi}\int_\Sigma \frac{f(Q)}{R^3}\, d\sigma \equiv -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\Sigma \frac{f(Q)}{R}\, d\sigma, \tag{21}$$

which is the limiting form, appropriate to the half-space, of the Poisson integral for the sphere.[17]

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[14] See [9], p. 140.

[15] The following developments remain valid if Σ is not bounded, so long as $f(Q)$ is sufficiently regular at infinity. See the remark at the end of the preceding section.

[16] See, for example, [15], p. 221.

[17] See [14], p. 240 *et seq.*

Next, we establish the singular solution $[S_0(Q, P)]$ of the thermoelastic field equations for the half-space. It suffices to take Q at the origin O, and we shall write $[S_0(P)]$ in place of $[S_0(O, P)]$. By (20), the temperature field belonging to $[S_0(P)]$ appears as,

$$T_0(P) = \frac{1}{2\pi}\frac{z}{r^3}, \tag{22}$$

provided r is the distance from the origin. $[S_0(P)]$ has rotational symmetry about the z-axis and, with reference to (10), (11), is generated by the stress functions,

$$\left.\begin{aligned} \phi &= -\frac{\beta}{2(1-\nu)}\frac{z}{r} + \beta \log (r + z), \\ \psi &= \frac{\beta}{2(1-\nu)}\frac{1}{r}, \end{aligned}\right\} \tag{23}$$

where

$$\beta = \frac{\alpha}{\pi}\mu(1 + \nu). \tag{24}$$

Clearly, ϕ and ψ satisfy (11) with $T = T_0(P)$. The corresponding displacements and stresses follow from (23) with the aid of (10), (3), and (2). If (ρ, γ, z) denote circular cylindrical coordinates, one reaches, after an elementary computation, the following cylindrical components of displacement and stress for $[S_0(P)]$:[18]

$$\left.\begin{aligned} &2\mu u_\rho = \frac{\beta\rho}{r(r+z)}, \qquad 2\mu u_z = -\frac{\beta}{r}; \\ &\tau_{\rho\rho} = -\frac{\beta}{r(r+z)}, \qquad \tau_{\gamma\gamma} = \beta\left[\frac{1}{r(r+z)} - \frac{z}{r^3}\right], \\ &\tau_{zz} = \tau_{z\rho} = 0. \end{aligned}\right\} \tag{25}$$

It is readily confirmed that $[S_0(P)]$ indeed meets the defining requirements (a), (b), (c), and (d), listed in the preceding section. We note parenthetically that[19]

$$[S_0(P)] = \frac{\partial}{\partial z}[S_0'(P)], \tag{26}$$

where $[S_0'(P)]$ is the solution corresponding to a (suitably normalized) heat source, located at the origin, in a medium occupying the entire space.[20] Thus $[S_0(P)]$ may also be interpreted as generated by a heat-doublet at O, whose axis coincides with the z-axis.

Let

$$\chi(P) = \int_\Sigma f(Q) \log (R + z)\, d\sigma. \tag{27}$$

[18]In axisymmetric solutions, the displacement u_γ and the stresses $\tau_{\gamma\rho}$, $\tau_{\gamma z}$, which vanish identically, will be omitted altogether.

[19]This identity is valid if both solutions appearing here are referred to cartesian or cylindrical coordinates.

[20]See [2], p. 74.

The harmonic function $\chi(P)$ is Boussinesq's [10] three-dimensional logarithmic potential for a disk in the shape of Σ, whose mass density is $f(Q)$, while[21]

$$U(P) = \chi_{,z} = \int_\Sigma \frac{f(Q)}{R}\,d\sigma \tag{28}$$

is the Newtonian potential of the same disk. From (21), (27) follows

$$T(P) = -\frac{1}{2\pi}\chi_{,zz}\,, \tag{29}$$

and according to (27), (23), (18), (10), and (7), the solution $[S(P)]$ to the thermoelastic problem under consideration, is characterized by the stress functions,

$$\left.\begin{aligned} \phi &= -\frac{\beta}{2(1-\nu)} z\chi_{,z} + \beta\chi, \\ \psi_x &= \psi_y = 0, \\ \psi_z &\equiv \psi = \frac{\beta}{2(1-\nu)}\chi_{,z}\,, \end{aligned}\right\} \tag{30}$$

where β is again given by (24).

Equations (30), together with (6), (3), and (2), yield the following expressions for the cartesian components of displacement and stress of $[S(P)]$, in terms of the potential χ:

$$\left.\begin{aligned} 2\mu u_x &= \beta\chi_{,x} & 2\mu u_y &= \beta\chi_{,y}\,, & 2\mu u_z &= -\beta\chi_{,z}\,; \\ \tau_{xx} &= -\beta\chi_{,yy}\,, & \tau_{yy} &= -\beta\chi_{,xx}\,, & \tau_{xy} &= \beta\chi_{,xy}\,, \\ \tau_{zx} &= \tau_{zy} = \tau_{zz} = 0. \end{aligned}\right\} \tag{31}$$

Inspection of (31) reveals the remarkable fact that the steady-state stress field induced by the surface distribution of temperature $f(Q)$ is plane[22] and parallel to the boundary, regardless of the shape of the region of exposure Σ and for every distribution of temperature on Σ. This rather severe degeneracy inherent in the *steady-state* problem for the half-space has a two-dimensional analog. In the two-dimensional treatment of the steady-state problem for the half-plane under an arbitrarily prescribed distribution of the edge temperature, all components of stress acting parallel to the half-plane vanish identically.[23]

In the case of rotational symmetry about the z-axis,[24] we obtain, on transforming (31) into cylindrical coordinates,

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[21]Subscripts attached to χ denote partial differentiation with respect to the arguments indicated.

[22]This could have been inferred directly from (25) and (18). Note that $\Phi \equiv -\beta\chi(x, y, z)$ is the Airy-function belonging to the plane stress distribution given in (31). Observe that the stress field depends on z.

[23]This result is a consequence of a general theorem which applies to any simply connected plane region; see [1], p. 428. Recall also that a temperature field $T(P)$ which is linear in x, y, z induces no stresses in a body of arbitrary shape: see, for example, [2], p. 9.

[24]Here Σ is necessarily a circle or a circular annulus centered at 0, and $f(Q)$ has polar symmetry with respect to the origin.

$$\left.\begin{aligned} 2\mu u_\rho &= \beta\chi_\rho\,, \qquad 2\mu u_z = -\beta\chi_z\,; \\ \tau_{\rho\rho} &= -\frac{\beta}{\rho}\chi_\rho\,, \qquad \tau_{\gamma\gamma} = -\beta\chi_{\rho\rho}\,, \qquad \tau_{z\rho} = \tau_{zz} = 0. \end{aligned}\right\} \tag{32}$$

Equations (29) and (31) or (32) reduce the solution of the present thermoelastic problem to the determination of the three-dimensional logarithmic potential defined in (27). This potential was first introduced by Boussinesq [10] in connection with his potential-theoretic treatment of the ordinary problem of the half-space under an arbitrary system of normal tractions applied to the plane boundary (in the absence of any temperature field).

Let $[S^*(P)]$ be the solution of Boussinesq's problem for the boundary conditions,

$$\tau_{zx}^*(x, y, 0) = \tau_{zy}^*(x, y, 0) = 0, \qquad \tau_{zz}^*(x, y, 0) = \mu\alpha f(Q). \tag{33}$$

A comparison of Boussinesq's representation[25] of $[S^*(P)]$ in terms of χ with the representation (31) of $[S(P)]$ at once supplies a useful connection between these two solutions; the subsequent formulas are valid only along the boundary $z = 0$:

$$\left.\begin{aligned} u_x &= \epsilon u_x^*\,, \qquad u_y = \epsilon u_y^*\,, \qquad u_z = \frac{1+\nu}{1-\nu}u_z^*\,; \\ \tau_{xx} &= \epsilon(\tau_{xx}^* - \tau_{zz}^*), \qquad \tau_{yy} = \epsilon(\tau_{yy}^* - \tau_{zz}^*), \qquad \tau_{xy} = \epsilon\tau_{xy}^*\,, \end{aligned}\right\} \tag{34}$$

where

$$\epsilon = \frac{2(1+\nu)}{1-2\nu}. \tag{35}$$

We note from (34) that the surface values of the displacement components and of the shearing stress in $[S(P)]$ are proportional to the corresponding values in $[S^*(P)]$.

The problem of Boussinesq characterized by the boundary conditions (33), was subsequently reconsidered by Lamb [16] and Terazawa [17] who confined their attention to the axisymmetric case; by an alternative method, they arrived at solutions in terms of definite integrals involving Bessel functions.[26] Love [7] returned to Boussinesq's integral form of the solution and arrived at closed representations for the required derivatives of the potential χ appropriate to a circular load region under a uniform or hemispherical distribution of the pressure, as well as to a rectangular region at constant pressure. In what follows we utilize Love's results to obtain closed exact solutions of the analogous steady-state thermoelastic problems for the half-space.

Circular region of exposure with uniform or hemispherical temperature distribution. Let $D + B$ again be the half-space $z \geq 0$ and let the region of exposure Σ, in particular, be the circle $0 \leq \rho \leq a$, $z = 0$. We first consider a uniform distribution of temperature over Σ or, in accordance with (19),

$$f(Q) = c \quad \text{for} \quad Q \quad \text{in} \quad \Sigma, \tag{36}$$

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[25] Boussinesq's results [10] are conveniently found in Eqs. (5), (7) of [7], p. 379.

[26] A complete bibliography of Boussinesq's problem is beyond the scope of the present paper. For further references, see [18], p. 243, and [1], p. 368.

where c is a constant. The solution to this axisymmetric problem is given in integral form by (29) and (32), χ being defined by (27) and (36). All of the partial derivatives of the logarithmic potential χ entering (32), were expressed by Love [7] in terms of complete and incomplete elliptic integrals of the first and second kind. The Newtonian potential[27] $U = \chi_z$ here coincides with the velocity potential of a uniform source disk in ideal incompressible fluid flow. In this context, Sadowsky and Sternberg [19] independently arrived at elliptic integral representations for U, and for the velocity components $U_\rho = \chi_{z\rho}$, $U_z = \chi_{zz}$; their results are of essentially the same form as those established previously by Love [7].

In preparation for an application of Love's work to the present problem, we introduce certain auxiliary position parameters and, for future reference, recall some relevant properties of elliptic integrals.[28] Thus let

$$r_1 = [(a - \rho)^2 + z^2]^{1/2}, \qquad r_2 = [(a + \rho)^2 + z^2]^{1/2}, \tag{37}$$

whence r_1 and r_2 are the respective distances of a point $P(\rho, z)$ from the points $M_1(a, 0)$

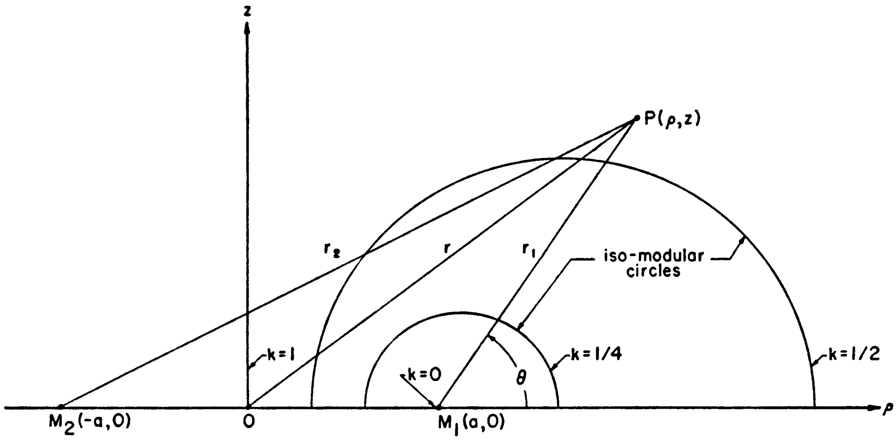

FIG. 1. Position parameters.

and $M_2(-a, 0)$, if all three points lie in the same meridional plane γ = constant (Fig. 1). We now define position parameters k, k' and Θ by means of

$$k = \frac{r_1}{r_2}, \qquad k' = (1 - k^2)^{1/2} = \frac{2}{r_2}(a\rho)^{1/2}, \tag{38}$$

and

$$\sin \Theta = \frac{z}{r_1}, \qquad \cos \Theta = \frac{\rho - a}{r_1}, \qquad (0 \le \Theta \le \pi), \tag{39}$$

so that Θ is the polar angle at M_1 shown in Fig. 1.

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[27]See Eq. (28).
[28]For a treatment of elliptic integrals, see, for example, [20], Ch. XXII.

Let $F(k, \Theta)$ and $E(k, \Theta)$, respectively, denote Legendre's normal form of the incomplete elliptic integral of the first and second kind, referred to the modulus k and to the argument Θ. Hence,

$$F(k, \Theta) = \int_0^\Theta \frac{dt}{(1 - k^2 \sin^2 t)^{1/2}}, \qquad E(k, \Theta) = \int_0^\Theta (1 - k^2 \sin^2 t)^{1/2}\, dt, \tag{40}$$

while

$$K = F(k, \pi/2), \qquad E = E(k, \pi/2), \tag{41}$$

are the corresponding complete elliptic integrals for the modulus k. We shall use K' and E' to designate the complete integrals of the first and second kind referred to the complementary modulus k'. The incomplete integrals satisfy the relations,

$$\left.\begin{aligned} F(k, -\Theta) &= -F(k, \Theta), \qquad F(k, \Theta + \pi) = 2K + F(k, \Theta), \\ E(k, -\Theta) &= -E(k, \Theta), \qquad E(k, \Theta + \pi) = 2E + E(k, \Theta), \end{aligned}\right\} \tag{42}$$

and we cite Legendre's identity,

$$KE' + EK' - KK' = \frac{\pi}{2}. \tag{43}$$

With a view toward shortening the results to be given subsequently, it is expedient to introduce an auxiliary function $H(k, \Theta)$, defined by

$$H(k, \Theta) = K'E(k, \Theta) + (E' - K')F(k, \Theta). \tag{44}$$

From (37), (38) follows,

$$\left(\frac{z}{a}\right)^2 + \left[\frac{\rho}{a} - \frac{1 + k^2}{(k')^2}\right]^2 = \frac{4k^2}{(k')^4}. \tag{45}$$

The parameters k and θ thus determine a (non-orthogonal) curvilinear coordinate system in the meridional half-plane γ = constant, $\rho \geq 0$. The coordinate lines k = constant are the family of "iso-modular" circles, centered on the ρ-axis at

$$\rho = \frac{(1 + k^2)a}{(k')^2}, \qquad z = 0, \tag{46}$$

and having the radius,

$$\delta = \frac{2ka}{(k')^2}. \tag{47}$$

These circles enclose M_1; they shrink to M_1 as $k \to 0$. The coordinate lines Θ = constant are the rays issuing from M_1 (see Fig. 1).

With reference to the foregoing notation, we now record the solution to the problem under consideration, obtained by substitution of Love's [7] formulas for the partial derivatives of χ into (29) and (32):[29]

$$T = \frac{c}{\pi}\left[H(k, \Theta) - \frac{z}{r_2} K'\right]; \tag{48}$$

[29]Recall footnote No. 21.

$$\left.\begin{aligned}
u_\rho &= \frac{c\beta}{2\mu\rho}\left[\pi a^2 - \frac{z}{r_2}(2a^2 + 2\rho^2 + z^2)K' + zr_2E' + (\rho^2 - a^2)H(k, \Theta)\right],\\
u_z &= \frac{c\beta}{\mu}\left[\frac{\rho^2 - a^2}{r_2}K' - r_2E' + zH(k, \Theta)\right];\\
\tau_{\rho\rho} &= \frac{c\beta}{\rho^2}\left[-\pi a^2 + \frac{z}{r_2}(2a^2 + 2\rho^2 + z^2)K' - zr_2E' + (a^2 - \rho^2)H(k, \Theta),\right]\\
\tau_{\gamma\gamma} &= \frac{c\beta}{\rho^2}\left[\pi a^2 - \frac{z}{r_2}(2a^2 + z^2)K' + zr_2E' - (a^2 + \rho^2)H(k, \Theta)\right],\\
\tau_{z\rho} &= \tau_{zz} = 0.
\end{aligned}\right\} \tag{49}$$

Of particular interest are the special values of the displacements and stresses on the axis of symmetry and along the boundary. On the z-axis, i.e., for $\rho = 0$, $z \geq 0$, we have from (37), (38), (39),

$$\left.\begin{aligned}
r_0 \equiv r_1 = r_2 = (a^2 + z^2)^{1/2}, \qquad k = 1, \qquad k' = 0,\\
\sin\Theta = \frac{z}{r_0}, \qquad \cos\Theta = -\frac{a}{r_0}.
\end{aligned}\right\} \tag{50}$$

Furthermore, with the aid of (50) and (41) to (44), there results, as $k \to 1$,

$$K \to \frac{\pi}{2}, \qquad E \to \frac{\pi}{2}, \qquad E' \to 1, \qquad H(k, \Theta) \to \pi\left(1 - \frac{z}{2r_0}\right). \tag{51}$$

Equations (48), (49), in view of (50), (51), yield the elementary formulas,

$$\left.\begin{aligned}
&T(0, z) = c\left(1 - \frac{z}{r_0}\right);\\
&u_\rho(0, z) = 0, \qquad u_z(0, z) = \frac{\pi\beta c}{\mu}(z - r_0);\\
&\tau_{\rho\rho}(0, z) = \tau_{\gamma\gamma}(0, z) = \pi\beta c\left(\frac{z}{r_0} - 1\right).
\end{aligned}\right\} \tag{52}$$

That $u_\rho(0, z)$ vanishes is verified by means of a Laurent expansion with respect to ρ, in a neighborhood of $\rho = 0$, of u_ρ in (49). The formulas for $\tau_{\rho\rho}$ and $\tau_{\gamma\gamma}$ in (52) are most conveniently obtained[30] by noting, on the basis of (32) and (29), that

$$2\tau_{\rho\rho}(0, z) = 2\tau_{\gamma\gamma}(0, z) = \beta(\chi_{zz})_{\rho=0} = -2\pi\beta T(0, z), \tag{53}$$

in view of the rotational symmetry and since χ is harmonic.

Next, we list the special values appropriate to the boundary $z = 0$, obtained from (48), (49) with the aid of (37) to (44).

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[30]This avoids a cumbersome limit process.

For $0 \leq \rho < a$:

$$\left.\begin{aligned} r_1 = a - \rho, \quad r_2 = a + \rho, \quad k = \frac{a - \rho}{a + \rho}, \quad k' = \frac{2(a\rho)^{1/2}}{a + \rho}, \\ \Theta = \pi, \quad H(k, \pi) = \pi; \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} &T(\rho, 0) = c, \\ &u_\rho(\rho, 0) = \frac{\pi c\beta\rho}{2\mu}, \quad u_z(\rho, 0) = \frac{c\beta}{\mu}[(\rho - a)K' - (\rho + a)E']; \\ &\tau_{\rho\rho}(\rho, 0) = \tau_{\gamma\gamma}(\rho, 0) = -\pi c\beta. \end{aligned}\right\} \tag{55}$$

For $a < \rho < \infty$:

$$\left.\begin{aligned} r_1 = \rho - a, \quad r_2 = \rho + a, \quad k = \frac{\rho - a}{\rho + a}, \quad k' = \frac{2(a\rho)^{1/2}}{\rho + a}, \\ \Theta = 0, \quad H(k, 0) = 0; \end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned} T(\rho, 0) = 0, \quad u_\rho(\rho, 0) = \frac{\pi c\beta a^2}{2\mu\rho}, \\ \tau_{\rho\rho}(\rho, 0) = -\tau_{\gamma\gamma}(\rho, 0) = -\frac{\pi c\beta a^2}{\rho^2}, \end{aligned}\right\} \tag{57}$$

while $u_z(\rho, 0)$ is again as in (55). With the aid of the Landen transformation,[31] we reach the alternative representation for $u_z(\rho, 0)$,

$$\left.\begin{aligned} &u_z(\rho, 0) = -\frac{2c\beta a}{\mu}E(\kappa), \qquad \kappa = \frac{\rho}{a} \qquad (0 \leq \rho \leq a), \\ &u_z(\rho, 0) = \frac{2c\beta\rho}{\mu\kappa}[(\kappa')^2K(\kappa) - E(\kappa)], \\ &\qquad\qquad \kappa = \frac{a}{\rho} \qquad (a \leq \rho < \infty). \end{aligned}\right\} \tag{58}$$

The special values along $z = 0$ agree with the general formulas (34), as is seen by comparing (55), (57) with the corresponding results[32] for the half-space under a uniform pressure applied to a circular load-region. We note that the normal displacement $u_z(\rho, 0)$ is analytic for $0 \leq \rho < \infty$; its graph is shown in Fig. 2. The radial displacement $u_\rho(\rho, 0)$ and the stress $\tau_{\rho\rho}(\rho, 0)$ are continuous throughout but suffer a discontinuity in their first derivative at $\rho = a$, i.e., at the edge of the region of exposure. On the other hand, the temperature $T(\rho, 0)$ and the stress $\tau_{\gamma\gamma}(\rho, 0)$ display a finite jump discontinuity at $\rho = a$. Indeed,

$$\left.\begin{aligned} \tau_{\gamma\gamma}(a+, 0) - \tau_{\gamma\gamma}(a-, 0) &= 2c\alpha(1 + \nu)\mu \\ &= -\alpha E[T(a+, 0) - T(a-, 0)], \end{aligned}\right\} \tag{59}$$

[31] See [20], p. 507.
[32] See Love [7].

if E, momentarily, denotes Young's modulus. This result is consistent with the plane-strain theory of steady-state thermal stresses.[33]

Finally, we examine the local behavior of the solution in the vicinity of the edge

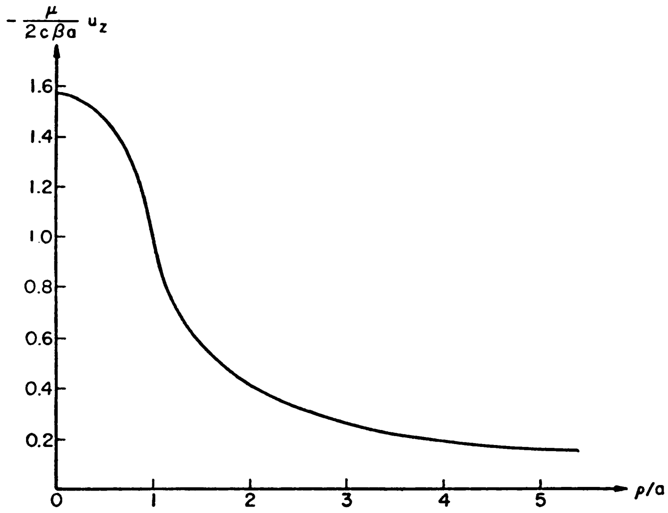

FIG. 2. Circular region of exposure at constant temperature. Normal displacement at the boundary.

$r_1 = 0$ of the region of exposure Σ. To this end, we note on the basis of (37) to (44) that $k \to 0$, $k' \to 1$ as $r_1 \to 0$, while[34]

$$K' = \log \frac{4}{k} + o(1), \qquad E' = 1 + o(1), \qquad H = \Theta + o(1), \tag{60}$$

where $o(1)$ denotes terms which tend to zero together with r_1 and k. From (60) and (48), (49) we infer

$$\left.\begin{aligned} T &= \frac{c\Theta}{\pi} + o(1), \\ u_\rho &= \frac{\pi c\beta a}{2\mu} + o(1), \qquad u_z = -\frac{2c\beta a}{\mu} + o(1), \\ \tau_{\rho\rho} &= -\pi c\beta + o(1), \qquad \tau_{\gamma\gamma} = c\beta(\pi - 2\Theta) + o(1), \end{aligned}\right\} \tag{61}$$

as r_1 approaches zero. This completes the discussion of the solution given in (48), (49). In view of the available tabulations[35] of all special functions entering this solution, its numerical evaluation presents no difficulties.

[33]See [1], Eq. (d), p. 428.
[34]See, also, [20], p. 521.
[35]Such as [21].

We now turn to a hemispherical temperature distribution over the circular region of exposure $0 \leq \rho \leq a$, $z = 0$. Thus, replace (36) with

$$f(Q) = c\left[1 - \left(\frac{\rho}{a}\right)^2\right]^{1/2} \quad \text{for} \quad Q \quad \text{in} \quad \Sigma. \tag{62}$$

The analogous Boussinesq problem, corresponding to a hemispherical pressure distribution, arises in the axisymmetric case of the Hertz contact problem.[36] Hertz [22] determined explicitly merely the normal displacement at the boundary. The solution was completed in closed form, and in terms of elementary functions, by Huber [23]. Love [7] reconsidered the problem and in this connection established all of the partial derivatives of the associated logarithmic potential χ which enter (29), (32).

Using Love's results, we may write down directly the solution to the steady-state thermoelastic problem governed by (62). Thus, let λ_1 be the positive root of the equation (for λ),

$$\frac{\rho^2}{a^2 + \lambda} + \frac{z^2}{\lambda} = 1, \tag{63}$$

and, for convenience, define η by

$$0 < \eta = \tan^{-1} \frac{a}{(\lambda_1)^{1/2}} \leq \pi/2. \tag{64}$$

The solution to the present problem becomes:

$$\left.\begin{aligned}
T &= \frac{cz}{a}\left(\frac{a}{(\lambda_1)^{1/2}} - \eta\right); \\
u_\rho &= \frac{\pi a^2 \beta c}{6\mu\rho}\left\{2 + \frac{3\rho^2 z}{a^3}\left[\frac{a}{(\lambda_1)^{1/2}} - \eta\right] - \frac{z}{(\lambda_1)^{1/2}}\left(3 - \frac{z^2}{\lambda_1}\right)\right\}, \\
u_z &= \frac{\pi a \beta c}{4\mu}\left[\left(\frac{\rho^2}{a^2} - \frac{2z^2}{a^2} - 2\right)\eta + \frac{2z^2}{a(\lambda_1)^{1/2}} - \frac{\rho^2(\lambda_1)^{1/2}}{a(a^2 + \lambda_1)}\right]; \\
\tau_{\rho\rho} &= \frac{\pi a^2 \beta c}{3\rho}\left\{-2 + \frac{3\rho^2 z}{a^3}\left[\eta - \frac{a}{(\lambda_1)^{1/2}}\right] + \frac{z}{(\lambda_1)^{1/2}}\left(3 - \frac{z^2}{\lambda_1}\right)\right\}, \\
\tau_{\gamma\gamma} &= \frac{\pi a^2 \beta c}{3\rho^2}\left\{2 + \frac{3\rho^2 z}{a^3}\left[\eta - \frac{a}{(\lambda_1)^{1/2}}\right] + \frac{z}{(\lambda_1)^{1/2}}\left(\frac{z^2}{\lambda_1} - 3\right)\right\}.
\end{aligned}\right\} \tag{65}$$

Rectangular region of exposure at constant temperature. In this section we consider the half-space $z \geq 0$ with a rectangular region of exposure Σ, defined by $-a \leq x \leq a$, $-b \leq y \leq b$, $z = 0$. Furthermore, let the temperature distribution be uniform over Σ, so that

$$f(Q) = c \quad \text{for} \quad Q \quad \text{in} \quad \Sigma, \tag{66}$$

in which c is again a constant. Love [7], in connection with the analogous Boussinesq problem, established in closed form the second partial derivatives of the potential χ

[36] See [1], art. 125.

defined in (27) for the density (66); he also indicated[37] the determination of the first partial derivatives which are needed in the computation of the displacements appropriate to the thermoelastic problem under consideration. In what follows we record merely the solution for the temperature and stress distributions. These follow directly from a substitution of Love's formulas for χ_{xx}, χ_{yy}, χ_{zz}, χ_{xy} into (29) and (31).

Let

$$\left.\begin{aligned} r_1 &= [(x-a)^2+(y-b)^2+z^2]^{1/2}, \qquad & r_2 &= [(x+a)^2+(y-b)^2+z^2]^{1/2}, \\ r_3 &= [(x+a)^2+(y+b)^2+z^2]^{1/2}, \qquad & r_4 &= [(x-a)^2+(y+b)^2+z^2]^{1/2}, \end{aligned}\right\} \tag{67}$$

whence r_1, r_2, r_3, r_4 are the distances of a point $P(x, y, z)$ from the corners of Σ. If we write c^{-1} and t^{-1} for arc-cosine and arc-tangent, respectively, the results take the form,

$$\left.\begin{aligned} T &= \frac{c}{2\pi}\Bigg[2\pi - c^{-1}\frac{(a-x)(b-y)}{[(a-x)^2+z^2]^{1/2}[(b-y)^2+z^2]^{1/2}} \\ &\qquad - c^{-1}\frac{(a-x)(b+y)}{[(a-x)^2+z^2]^{1/2}[(b+y)^2+z^2]^{1/2}} \\ &\qquad - c^{-1}\frac{(a+x)(b-y)}{[(a+x)^2+z^2]^{1/2}[(b-y)^2+z^2]^{1/2}} \\ &\qquad - c^{-1}\frac{(a+x)(b+y)}{[(a+x)^2+z^2]^{1/2}[(b+y)^2+z^2]^{1/2}}\Bigg], \\ \tau_{xx} &= \beta c\Bigg[t^{-1}\frac{(a-x)z}{(b-y)r_1} + t^{-1}\frac{(a+x)z}{(b-y)r_2} + t^{-1}\frac{(a+x)z}{(b+y)r_3} + t^{-1}\frac{(a-x)z}{(b+y)r_4} \\ &\qquad - t^{-1}\frac{a-x}{b-y} - t^{-1}\frac{a+x}{b-y} - t^{-1}\frac{a+x}{b+y} - t^{-1}\frac{a-x}{b+y}\Bigg], \\ \tau_{yy} &= \beta c\Bigg[t^{-1}\frac{(b-y)z}{(a-x)r_1} + t^{-1}\frac{(b-y)z}{(a+x)r_2} + t^{-1}\frac{(b+y)z}{(a+x)r_3} + t^{-1}\frac{(b+y)z}{(a-x)r_4} \\ &\qquad - t^{-1}\frac{b-y}{a-x} - t^{-1}\frac{b-y}{a+x} - t^{-1}\frac{b+y}{a+x} - t^{-1}\frac{b+y}{a-x}\Bigg], \\ \tau_{xy} &= \beta c \log\frac{(z+r_1)(z+r_3)}{(z+r_2)(z+r_4)}, \end{aligned}\right\} \tag{68}$$

where the ranges of the arc-cosines and arc-sines appearing in (68) are $[0, \pi]$ and $[-\pi/2, \pi/2]$, respectively.

We observe that $\tau_{xy} \to \infty$ as $r_i \to 0$ $(i = 1, 2, 3, 4)$, i.e., at the corners of Σ. These logarithmic singularities disappear if $T(x, y, 0)$ is made continuous.[38] It is of interest to examine the behavior of the solution (68) in the vicinity of one of the sides of the region of exposure, say the side $y = b$, $z = 0$. Let

$$\delta = [(y-b)^2+z^2]^{1/2}, \qquad \Theta = t^{-1}\frac{z}{y-b} \qquad (0 \le \Theta \le \pi). \tag{69}$$

[37]See the remark at the end of Sec. 4 of [7].

[38]See [7], Sec. 4.

Then, as $\delta \to 0$,

$$\left.\begin{aligned}
T &= \frac{c\Theta}{\pi} + o(1) \quad \text{for} \quad 0 \le x < a, \qquad T = 0 \quad \text{for} \quad a < x < \infty, \\
\tau_{xx} &= \beta c\left[\pi - 2\Theta - t^{-1}\frac{a+x}{2b} - t^{-1}\frac{a-x}{2b}\right] + o(1) \quad \text{for} \quad 0 \le x < a, \\
\tau_{xx} &= -\beta c\left[t^{-1}\frac{a+x}{2b} + t^{-1}\frac{a-x}{2b}\right] + o(1) \qquad \text{for} \quad a < x < \infty, \\
\tau_{yy} &= -\beta c\left[t^{-1}\frac{2b}{a+x} + t^{-1}\frac{2b}{a-x}\right] + o(1) \qquad \text{for} \quad 0 < x < \infty, \\
\tau_{xy} &= \frac{\beta c}{2}\log\frac{(a-x)^2[(x+a)^2+4b^2]}{(a+x)^2[(x-a)^2+4b^2]} + o(1) \qquad \text{for} \quad 0 < x < \infty.
\end{aligned}\right\} \tag{70}$$

From (70) follows, for $0 \le x < a$,

$$\tau_{xx}(x, b+, 0) - \tau_{xx}(x, b-, 0) = -\alpha E[T(x, b+, 0) - T(x, b-, 0)], \tag{71}$$

where E is Young's modulus, as is consistent with two-dimensional theory.[39]

Finally, consider

$$[S_L] = \lim_{a\to\infty} [S] \quad \text{for} \quad b \text{ fixed}, \tag{72}$$

in which $[S]$ is once more the solution (68). A trivial limit process yields for the limit-solution $[S_L]$:

$$T = \frac{c}{\pi}(\Theta_1 - \Theta_2), \qquad \tau_{xx} = -\alpha ET, \qquad \tau_{yy} = \tau_{xy} = 0, \tag{73}$$

where

$$0 \le \Theta_1 = t^{-1}\frac{z}{y-b} \le \pi, \qquad 0 \le \Theta_2 = t^{-1}\frac{z}{y+b} \le \pi. \tag{74}$$

We identify[40] $[S_L]$ as the plane-strain solution associated with the steady-state thermoelastic problem for a half-plane $x = 0$, $z \ge 0$, if the edge $z = 0$ is subjected to the temperature distribution,

$$T(y, 0) = c \quad \text{for} \quad |y| < b, \qquad T(y, 0) = 0 \quad \text{for} \quad |y| > b. \tag{75}$$

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