—NOTES—

ON THE CONDITIONS OF VALIDITY OF RIEMANN'S METHOD OF INTEGRATION*

By AUREL WINTNER (*The Johns Hopkins University*)

1. Introduction. The traditional treatment of Riemann's classical formula depends on certain additional assumptions which are not necessary for the existence of a solution (Sec. 2) but, being based on the adjoint equation (Sec. 3), are necessary for the customary introduction of Riemann's Green function.

The purpose of this note is to point out the resulting complication and to show that it can be overcome by the application of a simple device in the proof. The main result is that italicized in Sec. 8.

It will be sufficient to consider only one of the standard cases, the case in which the boundary values of the unknown function itself are assigned along a path consisting of two characteristics which meet at a point. In fact, it will be clear from the nature of the arguments to be applied that all considerations remain valid for the case in which the data are, for instance, Cauchy data proper, the case in which the unknown function and its normal derivative are assigned along a continuously differentiable path on which no direction is a characteristic direction.

2. Picard's theorem. On the closed rectangle $0 \leqq x \leqq \xi$, $0 \leqq y \leqq \eta$, which will be denoted by $Q = Q(\xi, \eta)$, consider Riemann's hyperbolic partial differential equation

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \tag{1}$$

with the boundary data

$$u(x, \eta) = \varphi(x), \qquad u(\xi, y) = \psi(y), \tag{2}$$

where $\varphi(x)$ $(0 \leqq x \leqq \xi)$ and $\psi(y)$ $(0 \leqq y \leqq \eta)$ are given functions satisfying

$$\varphi(\xi) = \psi(\eta). \tag{3}$$

The appropriate conditions, to be imposed on (a, b, c) and (φ, ψ), respectively, are as follows: (i) the coefficient functions $a(x, y)$, $b(x, y)$, $c(x, y)$ are continuous on Q and (ii) the boundary functions $\varphi(x)$, $\psi(y)$ are continuously differentiable. In fact, the situation is as follows.

The pair conditions (i)-(ii) assures that there exists on the rectangle Q a unique function $u = u(x, y)$ having the following properties:

$$u_x\ ,\ u_y \text{ and } u_{xy}(= u_{yx}) \text{ exist and are continuous,} \tag{4}$$

the relation (1) is an identity on Q and the two equations (2) are identities on the sides $y = \eta$, $x = \xi$ of Q.

This will be referred to as Picard's theorem. It can be proved by the method of

---

*Received February 3, 1956.

successive approximations and is the precise formulation of what is actually proved in Picard's writings on the subject.[1]

3. The formal adjoint. In contrast to what is supplied by Picard's theorem, Riemann's method of integration replaces Eq. (1) by its adjoint,

$$v_{xy} - [a(x, y)v]_x - [b(x, y)v]_y + c(x, y)v = 0, \tag{1 bis}$$

to which certain boundary data

$$v(x, \eta) = \rho(x), \qquad v(\xi, y) = \sigma(y) \tag{2 bis}$$

satisfying

$$\rho(\xi) = \sigma(\eta) \tag{3 bis}$$

are assigned.

The traditional presentation of Riemann's method[2] depends on an application of Picard's theorem to the case in which the Eq. (1) and the boundary condition (2) are replaced by the adjoint equation (1 *bis*) and a certain boundary condition (2 *bis*), respectively. A moment's reflection shows, however, that this application is inadmissible under the assumptions (i)-(ii) placed on (1)-(2) by Picard's theorem. For, since (i) merely assumes the continuity of $a(x, y)$ and $b(x, y)$, the derivatives $a_x(x, y)$, $b_y(x, y)$ need not exist (or, if they do, they need not be continuous) and so Eq. (1 *bis*) cannot in general be expanded into

$$v_{xy} + A(x, y)v_x + B(x, y)v_y + C(x, y)v = 0$$

with certain functions A, B, C, still less with continuous functions A, B, C, as it is required by (i) when Eq. (1 *bis*) is identified with Eq. (1).

4. Methodical remarks. The purpose of this note is to point out an easy way out of the implications of this predicament.

For the case of the elliptic analogues of the hyperbolic equation (1), a similar difficulty was circumvented by Lichtenstein,[3] by using an "integrated form" of the adjoint. In the hyperbolic case at hand, a simple and direct approach, which makes explicit a function space allowable for the adjoint under the original assumption (i) on Eq. (1), can be obtained.

Today, Lichtenstein's result can be interpreted as a manifestation of L. Schwartz's distribution theory[4]. From the point of view of the theory of distributions, the success of the explicit approach in the hyperbolic case will center around the following formal fact. If the set-function $I^q(f)$ (of the set q) is the integral of a continuous $f = f(x, y)$ over a rectangle $q = (x_1 \leqq x \leqq x_2\ , y_1 \leqq y \leqq y_2)$, and if f is one of the (continuous)

---

[1]See, e.g., E. Picard, *Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique*, 1927, pp. 123-134. For an extension to the case of certain nonlinear equations, a case the treatment of which is not based on the method of successive approximations but on that of equicontinuous functions, see P. Hartman and A. Wintner, Amer. J. of Math. **74**, 836-843 (1952)

[2]See, e.g. E. Picard, *op. cit.*, pp. 147-151 or G. Darboux, *Leçons sur la théorie générale des surfaces*, **2**, 71-81 (1889). The literature consulted on the extension of the classical conditions on Riemann's method (see, e.g., G. S. S. Ludford, J. of Ratl. Mech. and Anal. **3**, 77-88 (1954), where further references are given) does not go into the problems of the adjoint considered in this paper

[3]L. Lichtenstein, Bull. Acad. Polon. des Sciences **1931**, 571-598

[4]L. Schwartz, *Théorie des distributions*, vol. 1-2, 1950-1951

derivatives z_x, z_y, z_{xy} of a function $z = z(x, y)$, then the evaluation of $I^a(f)$ involves no differentiation of $z(x, y)$ in any of the three cases $f = z_x, z_y, z_{xy}$ (it involves integrations in the first two of the three cases).

5. Reformulation of the adjoint problem. Suppose that the coefficient functions a, b, c of Eq. (1) are subject only to the continuity assumption (i) on Q. Then Eq. (1 *bis*), as it stands, is meaningless in general. But it appears in a meaningful form if it is integrated over the rectangle having (x, y) and (ξ, η) as opposite vertices, where (x, y) is any point of the given rectangle $Q = (0 \leqq x \leqq \xi, 0 \leqq y \leqq \eta)$. In fact, the formal result of this integration is

$$v(\xi, \eta) - v(\xi, y) - v(x, \eta) + v(x, y)$$
$$- \int_y^\eta a(s, t)v(s, t) \Big|_{s=x}^{s=\xi} dt - \int_x^\xi b(s, t)v(s, t) \Big|_{t=y}^{t=\eta} ds + \int_x^\xi \int_y^\eta c(s, t)v(s, t)\, ds\, dt = 0.$$

In view of the boundary data (2 *bis*) and the condition (3 *bis*), this can be written in the form

$$v(x, y) = -\rho(\xi) + \rho(x) + \sigma(y) - \int_x^\xi \int_y^\eta c(s, t)v(s, t)\, ds\, dt$$
$$+ \int_y^\eta \{a(\xi, t)v(\xi, t) - a(x, t)v(x, t)\}\, dt + \int_x^\xi \{b(s, \eta)v(s, \eta) - b(s, y)v(s, y)\}\, ds,$$

[where $-\rho(\xi)$ can be replaced by $-\sigma(\eta)$]. Let the latter formula for $v(x, y)$, a formula which is an integral equation for the unknown v, with a, b, c and ρ, σ as data, be referred to as Eq. (1*).

Under appropriate assumptions of differentiability, Eq. (1*) is equivalent to Eq. (1 *bis*) and (2 *bis*) together. But Eq. (1*) is meaningful under the following pair of assumptions also: (i*) the continuity assumption (i) of Sec. 2 for $a(x, y)$, $b(x, y)$ $c(x, y)$ on Q and (ii*) the continuity of $\rho(x)$, $\sigma(y)$ on the respective intervals $0 \leqq x \leqq \xi$, $0 \leqq y \leqq \eta$ [note that (ii*) requires of ρ, σ less than (ii) in Sec. 2 requires of φ, ψ].

6. The dual of Picard's theorem. Corresponding to the circumstance that the formulation (1*) of the adjoint problem is free of any differentiation, it is natural to extend the solution class, defined by condition (4) in Picard's theorem, as follows: a "solution" $v = v(x, y)$ should mean any function defined on Q in such a way that

$$v(x, y) \text{ is continuous} \tag{4*}$$

on Q and satisfies the Eq. (1*) as an identity on Q. The fundamental fact, representing the true dual of Picard's theorem, can then be formulated as the following theorem.

If (1) and (2) are replaced by (1*) and, at the same time, (i), (ii) and (4) are replaced by (i*), (ii*) and (4*), respectively [which means that (i) is retained but both (ii) and (4) are extended to the assumption of mere continuity], then there exists on Q exactly one solution $v(x, y)$.

The proof of this dual of Picard's theorem depends on successive approximations. To this end, let $v_0(x, y) \equiv 0$ on Q and, if $v_n(x, y)$ has already been defined on Q as a continuous function, let $v_{n+1}(x, y)$ denote the function which results if v is replaced by v_n in the sum which represents the expression on the right of Eq. (4*). It then follows that $v_1(x, y)$, $v_2(x, y)$, $\cdots$ is a sequence of continuous functions which tend to a limit function uniformly on Q and that, if $v(x, y)$ denotes the limit function, then $v(x, y)$ is a solution of Eq. (4*). The details are of a routine nature and will therefore be omitted. The assumption that there exist two distinct solutions $v(x, y)$ leads, again by successive approximations, to a contradiction in the usual way.

7. The reciprocity relation of the Green functions. Under assumption (i) [which is identical with assumption (i*)] choose φ, ψ and ρ, σ as follows:

$$\varphi(x) = \exp \int_x^\xi -b(s, \eta)\, ds, \qquad \psi(y) = \exp \int_y^\eta -a(\xi, t)\, dt \tag{5}$$

and

$$\rho(x) = \exp \int_x^\xi b(s, \eta)\, ds, \qquad \sigma(y) = \exp \int_y^\eta a(\xi, t)\, dt \tag{5 bis}$$

(so that $\rho = 1/\varphi$, $\sigma = 1/\psi$). Then conditions (ii), (3) and (ii*), (3 *bis*) are satisfied. Hence both of the above theorems (those of Sec. 2 and Sec. 5) are applicable if a, b, c are just continuous. With reference to the rectangle Q, having $(0, 0)$ and (ξ, η) as opposite vertices, let $G(x, y) = G(x, y; \xi, \eta)$ denote that solution $u(x, y)$, subject to condition (4), of Eq. (1) which belongs to the Cauchy data (5), and let $H(x, y) = H(x, y; \xi, \eta)$ denote that solution $v(x, y)$, subject to condition (4*), of Eq. (1*) which belongs to the Cauchy data (5 *bis*). Then, if (x', y'), (x'', y'') is any pair of points contained in the rectangle Q, Riemann's reciprocity theorem

$$H(x', y'; x'', y'') = G(x'', y''; x', y') \tag{6}$$

is valid.

Since a, b (and c) are now just continuous, the classical proof of the symmetry relation[5], a proof based on Green's formula, fails to apply. One can however conclude as follows.

It is readily seen that the successive approximations leading to $u = G(x', y'; x'', y'')$ and $v = H(x', y'; x'', y'')$ are uniform in the points (x', y'), (x'', y'') of Q and in the index $k(= 1, 2, \cdots)$ together, if (a_1, b_1, c_1), (a_2, b_2, c_2), $\cdots$, (a_k, b_k, c_k), $\cdots$ are sequences of functions which are continuous on Q and tend to (a, b, c) uniformly on Q. Hence, the relation (6) is true for (a, b, c) if it is true for every (a_k, b_k, c_k). Since it is true in the classical case, it follows for the case of just continuous coefficient functions (a, b, c); in fact, $a_k(x, y)$, $b_k(x, y)$, $c_k(x, y)$ can be chosen to be polynomials.

8. The extended validity of Riemann's formula. Without any reference to the relation (6) (which will not be used directly), consider again the problem of Sec. 2, represented by Eqs. (1), (2) and (3), with (i) and (ii) as assumptions. According to Picard's theorem, this problem has on Q a unique solution,

$$u = u(x, y; a, b, c, \varphi, \psi) \tag{7}$$

satisfying condition (4). Riemann's classical formula represents this unique solution (7) in terms of the Green function $H(x, y; \xi, \eta)$ of the adjoint equation. But the classical definition of H is meaningless under the present assumptions (see Sec. 3). On the other hand, if the continuous function $H(x, y; \xi, \eta)$ is defined, not in the traditional manner, but in the way specified in Sec. 7 (as supplied by the existence and uniqueness theorem

[5]See, e.g., G. Darboux, *op. cit.*, p. 81

of Sec. 6), then *Riemann's integral representation of the solution* (7) *remains valid under the assumptions,* (i) *and* (ii), *of Picard's theorem alone.*

In fact, Riemann's integral representation[6] of the solution (7) contains only the following elements: (α) the boundary data $\varphi(x)$, $\psi(y)$ [which are subject to condition (3)] and their first derivatives and (β) the function $H(x, y; \xi, \eta)$ and their *first* derivatives on the boundary; cf. (5), (5 *bis*). But item (β) does not involve the *second* derivatives of the function H (derivatives which do in general exist). On the other hand, not only the functions $\varphi(x)$, $\psi(y)$ but also their first derivatives, introduced by item (α), are controlled by assumption (ii). Hence it is clear that the italicized assertion, concerning the general validity of Riemann's formula, can be concluded by the same argument (this time by approximating *$a(x, y)$, $b(x, y)$ and $\varphi(x)$, $\psi(y)$ as well as $d\varphi(x)/dx$, $d\psi(y)/dy$* by sequences of smooth functions) which was applied in Sec. 7.

NOTE ON THE AERODYNAMIC HEATING OF AN OSCILLATING INSULATED SURFACE*

BY STEPHEN H. MASLEN AND SIMON OSTRACH

(*Lewis Flight Propulsion Laboratory, National Advisory Committee for Aeronautics, Cleveland, Ohio*)

The effect of disturbing the thermal equilibrium of an oscillating conducting surface and its surroundings by changing the isothermal surface temperature at a given time was investigated in Ref. [1]. It was shown therein that the heat transfer and the thermal state of the fluid associated with the oscillating surface can be significantly different from that for conduction from a stationary surface with the same initial temperature difference. To complete this study it is appropriate to investigate the effect of insulating the surface at a given time on the equilibrium state.

Accordingly, consideration is given herein to a doubly infinite plane surface which is oscillating axially (i.e., longitudinally) in a viscous and heat-conducting fluid. It is assumed that sufficient time has elapsed so that an equilibrium state exists in which a periodic motion of the fluid has been established, and the heat obtained by viscous dissipation is all conducted through the surface so that the temperature does not increase indefinitely with time. In this state the fluid velocity is given by [2]

$$u(y, t) = U \exp [-(n/2\nu)^{1/2}y] \cos [nt - (n/2\nu)^{1/2}y] \tag{1}$$

and the temperature is [1]

$$T_e = T_\infty - \frac{U^2 Pr}{4c_p} \Big\{ \exp [-(2n/\nu)^{1/2}y] - \frac{1}{2 - Pr} [\exp \{-(n/\alpha)^{1/2}y\} \cos \{2nt - (n/\alpha)^{1/2}y\} - \exp \{-(2n/\nu)^{1/2}y\} \cos \{2nt - (2n/\nu)^{1/2}y\}] \Big\}, \tag{2}$$

where u is the fluid velocity component parallel to the surface, y is the coordinate normal to the surface, t denotes time, U and n are the amplitude and frequency of the surface

[6]See, e.g., G. Darboux, *op. cit.*, formula (16) on p. 80

*Received February 20, 1956.