A UNIQUENESS THEOREM FOR THE COUPLED THERMOELASTIC PROBLEM*

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1. Introduction. In the computation of thermal stresses in an elastic solid it is customary to compute first the temperature distribution by use of the Fourier heat conduction equation and then to determine the resulting thermal stresses according to the usual thermoelastic theory. Although this procedure is sufficiently accurate for a large class of problems, it is approximate since the Fourier heat conduction equation is an energy balance which neglects the interconvertibility of mechanical and thermal energy. If this possibility is included in the analysis, the energy balance equation contains both thermal and mechanical terms and the thermal and thermoelastic problems are coupled; the temperature and stress distributions must be determined simultaneously rather than consecutively. When the non-uniform temperature distribution is primarily due to heat supplied to the body from external sources, the mechanical coupling term in the energy balance may be neglected in comparison with the thermal terms; when the temperature differences are due solely to the deformations of the body, as in a study of thermoelastic damping for example, then the coupled problem must be considered.

The coupled nature of the thermal-thermoelastic problem was already known by Duhamel [1]; a derivation of the governing equations utilizing thermodynamic principles[1] was given by Voigt [2] who presents a linear theory valid for sufficiently small temperature changes, displacements and displacement gradients. The magnitude of permissible temperature changes depends upon the mean absolute temperature of the solid and the degree of temperature dependence of its elastic and thermal properties, while the restrictions on the displacements and their gradients are those required in the linear theory of elasticity.

In this note, a uniqueness theorem is presented for the coupled problem formulated by Voigt for the case of an isotropic elastic solid. The notation used is as follows: σ_{ij}, e_{ij}, u_i are components of the stress, strain and displacement tensors referred to a cartesian coordinate system $x_i (i, j = 1, 2, 3)$, λ, μ are Lamé's constants, ρ is the density, α is the coefficient of thermal expansion and $m = (3\lambda + 2\mu)\alpha$. T is the absolute temperature, T_0 is a reference temperature chosen so that $|(T - T_0)/T_0| \ll 1$ throughout the body, K is the thermal conductivity, c is the specific heat for processes with invariant strain tensor. The comma notation is used for derivatives with respect to space variables, superposed dots for derivatives with respect to the time, t, δ_{ij} is the Kronecker delta and the summation convention is employed.

2. Theorem. Given a regular region[2] of space $V + S$ with boundary S. Then there exists at most one set of single-valued functions $\sigma_{ij}(P, t)$ and $e_{ij}(P, t)$ of class $C^{(1)}$, $u_i(P, t)$ and $T(P, t)$ of class $C^{(2)}$ for $P(x_1, x_2, x_3)$ in $V + S$, $t \geq 0$ which satisfy the following equations for P in V, $t > 0$,

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[1]It should be noted here that these thermodynamic principles, as applied to deformable media, have not yet been put on a firm logical foundation, see [3], pp. 170–171. See also [7], which came to the author's attention after submission of the manuscript, for a comprehensive treatment of this subject from the viewpoint of irreversible thermodynamics and a physical interpretation of the integral obtained in Eq. (15) below.

[2]As defined in [4], p. 113.

$$KT_{,kk} = \rho c T^{\cdot} + mT_0 e_{kk}^{\cdot}, \tag{1}$$

$$\sigma_{ij,j} = \rho u_i^{\cdot\cdot}, \tag{2}$$

the following equations for P in $V + S$, $t \geq 0$,

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{3}$$

$$\sigma_{ij} = \delta_{ij}\lambda e_{kk} + 2\mu e_{ij} - \delta_{ij} mT, \tag{4}$$

the following equations for P on S, $t > 0$,

$$T = F^{(1)}(P, t), \tag{5}$$

$$u_i = G_i^{(1)}(P, t), \tag{6}$$

and the following equations for P in V, $t = 0$

$$T = F^{(2)}(P), \tag{7}$$

$$u_i = G_i^{(2)}(P), \tag{8}$$

$$u_i^{\cdot} = G_i^{(3)}(P), \tag{9}$$

where the constants K, c, λ, μ, m and T_0 are all positive.

Proof. Let there be two such sets of functions, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$ etc., and let $\sigma_{ij}^{*} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, $e_{ij}^{*} = e_{ij}^{(1)} - e_{ij}^{(2)}$, etc. By virtue of the linearity of the problem, it is clear that these difference functions will also satisfy Eqs. (1)-(4) and the homogeneous counterparts of Eqs. (5)-(9). In the calculations which follow the stars will be omitted from the designations of the difference functions. Consider the integral

$$\int_V \sigma_{ij} e_{ij}^{\cdot}\, dV = \int_V \sigma_{ij} u_{i,j}^{\cdot}\, dV = \int_V [(\sigma_{ij} u_i^{\cdot})_{,j} - (\sigma_{ij,j} u_i^{\cdot})]\, dV, \tag{10}$$

where Eq. (3) and the symmetry of the stress tensor have been utilized. By use of the divergence theorem and the homogeneous form of Eq. (6),

$$\int_V (\sigma_{ij} u_i^{\cdot})_{,j}\, dV = \int_S \sigma_{ij} u_i^{\cdot} n_j\, dS = 0.$$

Also, from Eq. (2),

$$\int_V \sigma_{ij,j} u_i^{\cdot}\, dV = \int_V \rho u_i^{\cdot\cdot} u_i^{\cdot}\, dV = \int_V \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u_i^{\cdot} u_i^{\cdot}\right) dV.$$

Therefore, Eq. (10) becomes,

$$\int_V \left[\sigma_{ij} e_{ij}^{\cdot} + \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u_i^{\cdot} u_i^{\cdot}\right)\right] dV = 0. \tag{11}$$

From Eq. (4) it is found that

$$\sigma_{ij} e_{ij}^{\cdot} = \lambda e_{mm} e_{kk}^{\cdot} + 2\mu e_{ij} e_{ij}^{\cdot} - mTe_{kk}^{\cdot} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\lambda e_{kk}^2 + \mu e_{ij} e_{ij}\right) - mTe_{kk}^{\cdot}$$

so that Eq. (11) may be rewritten in the form

$$\int_V \left[\frac{\partial}{\partial t}\left(\tfrac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \tfrac{1}{2}\rho u_i^{\cdot 2}\right) - mTe_{kk}^{\cdot}\right] dV = 0. \tag{12}$$

The following identity is readily derived by use of the divergence theorem

$$\int_V TT_{,kk}\, dV + \int_V T_{,k}T_{,k}\, dV = \int_V (TT_{,k})_{,k}\, dV = \int_S TT_{,k}n_k\, dS.$$

Substitution for $T_{,kk}$ from Eq. (1) in the above and use of the homogeneous form of Eq. (5) yields

$$\int_V T(\rho c T^{\cdot} + mT_0 e_{kk}^{\cdot})\, dV + K\int_V T_{,k}T_{,k}\, dV = 0$$

or

$$T_0 \int_V mTe_{kk}^{\cdot}\, dV = \frac{-\rho c}{2}\int_V \frac{\partial}{\partial t} T^2\, dV - K\int_V T_{,k}T_{,k}\, dV. \tag{13}$$

Substitution of Eq. (13) into Eq. (12) and interchange of the order of differentiation and integration then yields,

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \frac{1}{2}\rho u_i^{\cdot 2} + \frac{\rho c}{2T_0}T^2\right) dV = \frac{-K}{T_0}\int_V T_{,k}T_{,k}\, dV \leq 0. \tag{14}$$

The integral on the left hand side of the above equation is initially zero since the difference functions satisfy homogeneous initial conditions. By the inequality derived above, however, this integral either decreases (and therefore becomes negative) or remains equal to zero. Since its integral is the sum of squares, however, only the latter alternative is possible. That is

$$\int_V \left(\frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \frac{1}{2}\rho u_i^{\cdot 2} + \frac{\rho c}{2T_0}T^2\right) dV = 0, \qquad t \geq 0. \tag{15}$$

It follows from Eq. (15) that the difference functions are identically zero throughout the body and for all time and the theorem is proved.

3. Remarks. As noted in the Introduction, omission of the last term in Eq. (1) reduces the problem defined by Eqs. (1)-(9) to a heat conduction problem and a thermoelastic problem which are uncoupled. In this case, the right hand side of Eq. (13) is equal to zero and this equation may be used directly to prove the uniqueness of the temperature distribution. It follows then that the difference temperature distribution, T, in Eq. (12) is zero and that equation may be used in the usual manner to prove the uniqueness of the solution of the thermoelastic problem.

The above theorem may be generalized readily to include the most general linear thermal and mechanical boundary conditions[3]. Also the analytical restrictions on the solutions may be considerably lightened. These generalizations have not been considered here since the primary purpose of this note is to indicate how methods of proving uniqueness of the uncoupled heat conduction and elastic problems may be combined

[3]More general mechanical boundary conditions which are sufficient to prove uniqueness are, in fact, implicit in the second integral of the equation following Eq. (10).

to prove uniqueness of the coupled problem. The reader is therefore referred to other sources, e.g., [5, 6], for more detailed treatments of the uncoupled problems.

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References

1. J. M. C. Duhamel, *Second mémoire sur les phénomènes thermo-mécaniques*, J. de l'École Polytech. **15**, 1-57 (1837)
2. W. Voigt, *Lehrbuch der Kristallphysik*, Teubner, 1910
3. C. Truesdell, *The mechanical foundations of elasticity and fluid dynamics*, J. Rational Mechanics and Analysis **1**, 125-300 (1952)
4. O. D. Kellogg, *Foundations of potential theory*, J. Springer, Berlin, 1929
5. G. Doetsch, *Les équations aux derivées partielles du type parabolique*, L'Enseignement Mathematique **35**, 43 (1936)
6. E. Sternberg and R. Eubanks, *On the concept of concentrated loads and an extension of the uniqueness theorem in the linear theory of elasticity*, J. Rational Mechanics and Analysis **4**, 135-168 (1955)
7. M. A. Biot, *Thermoelasticity and irreversible thermodynamics*, J. Appl. Phys. **27**, 240–254 (1956)

THE MOTION OF A THERMOELASTIC SOLID*

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Introduction. It is the purpose of this paper to set down formally the relevant equations of thermo-elasticity within the approximation of the theory of elasticity of infinitesimal displacements and displacement derivatives. It is not claimed that the following equations of thermo-elasticity are original; Duhamel [1] and Neumann [2] derived similar equations many years ago, but due to the fact that they did not derive their equations from thermodynamic considerations and also that elasticians generally are not aware of the role of thermodynamics in their field, it was felt that a derivation of the general equations and application to a particular problem were in order. The present work is a refinement of Refs. [3] and [4] and the application was inspired by the work of Synge [5] in connection with the motion of a viscous, heat conducting fluid. The author is indebted to the reviewer for his extensive commentary which assisted materially in the revised version of this paper.

Analysis. For the case of small displacements and displacement derivatives, the momentum and energy (First Law) equations for a continuous, homogeneous medium may be written as

Momentum
$$\rho \frac{\partial^2 u_i}{\partial t^2} = \tau_{ki,k} , \tag{1}$$

Energy
$$\rho \frac{\partial U}{\partial t} = K_{mn} T_{,mn} + \tau_{mn} \frac{\partial u_{n,m}}{\partial t} , \tag{2}$$

where ρ is density, u_i is displacement vector, t is time coordinate, τ_{ki} is stress tensor, U is specific internal energy, K_{mn} is thermal conductivity tensor, and T is the temperature. The subscript notation is that of cartesian tensorial form and subscripts following a

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