THE RECTIFICATION OF NON-GAUSSIAN NOISE*

BY

JAMES A. MULLEN (*Raytheon Manufacturing Company, Waltham, Massachusetts*)

AND

DAVID MIDDLETON (*Concord, Massachusetts*)

Section I. Introduction. This paper considers the effects of a general class of non-gaussian random processes in an *a-m* receiving system. Analysis of the effects of non-gaussian noise is needed both for use in problems where the noise is known not to be gaussian, e.g. some kinds of radar clutter and atmospheric static, and to indicate in uncertain cases how critical the assumption of normal statistics may be.

The probability densities of a generalized Poisson process and a new approximating series for the densities are presented in Sec. II, with discussion of and results for the rectification problem following in Sec. III.

The asymptotic approximation to the probability distributions, using derivatives with respect to the second moments, is new in its general form, although foreshadowed by some special results obtained earlier by Edgeworth [1] and Pearson [2]. The problem of rectification of non-gaussian noise does not appear to have been attacked with reasonable generality before. A number of papers have dealt with the deviation of rectifier outputs from normal statistics when the input is normally distributed [3, 4, 5] and some results are available for non-gaussian noise in quadratic detectors [6] and for very narrow post detector filters [7]. We have obtained expressions for the output covariance function of a half wave νth-law rectifier with a sine wave carrier and non-gaussian noise input. Explicit results are presented for linear and quadratic detectors. A qualitative discussion of the behavior of rectified non-gaussian noise for general values of ν is also included.

In general, the work has shown that the difference between the effects of gaussian and non-gaussian noise of the same input power is fairly small unless the signal is weak or the noise is strongly non-gaussian. Usually non-gaussian noise produces a greater output intensity and a broader output spectrum than gaussian noise with the same input power and input spectrum.

Section II. Probability densities. 2.1 Introduction. The probability concepts and notation used here are briefly presented in this section. Probability densities are used rather than cumulative distribution functions; in the usual way, for example, one writes $W_2(y_1, y_2; t_2 - t_1)\, dy_1\, dy_2$ = the joint probability for a stationary random process that y will lie in the interval $(y_1, y_1 + dy_1)$ at time t_1, and in the interval $(y_2, y_2 + dy_2)$ at time t_2.

The distribution can also be characterized by its moments (whenever these exist), e.g.(*)

$$\mu_{mn}(t) = \langle y_1^m y_2^n \rangle = \iint_{-\infty}^{\infty} y_1^m y_2^n W_2(y_1, y_2; t)\, dy_1\, dy_2 . \tag{1}$$

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(*)The angle brackets here and subsequently denote the statistical average.

The Fourier transform of the probability density is the characteristic function,

$$F_{2y}(\xi_1, \xi_2; t) = \langle \exp(i\xi_1 y_1 + i\xi_2 y_2) \rangle, \tag{2}$$

whose power series expansion generates the moments,

$$F_2(\xi_1, \xi_2; t) = \sum_{m,n=0}^{\infty} \frac{(i\xi_1)^m (i\xi_2)^n}{m!\,n!} \mu_{mn}(t), \tag{3}$$

(again, whenever $\mu_{mn}(t)$ exist). An equally useful set of parameters are the semi-invariants, $\lambda_{mn}(t)$, defined by

$$\log F_2(\xi_1, \xi_2; t) = \sum_{m,n=1}^{\infty} \frac{(i\xi_1)^m (i\xi_2)^n}{m!\,n!} \lambda_{mn}(t). \tag{4}$$

2.2 The Poisson ensemble. The non-gaussian random processes which we shall use are Poisson ensembles, i.e. they are composed of sums of independent variables with common distributions and uniformly distributed times of occurrence, t_j', viz.,

$$V(t; K, \{t_j'\}) = \sum_{j=1}^{K} v_j(t - t_j'). \tag{5}$$

Physically, the v_j represent pulses produced randomly by a noise source. The above assumptions imply that the number of pulses, K, occurring in a long time interval, T, is a random variable with a Poisson distribution [8] so that K, as well as the set of t_j', is an ensemble parameter.

The characteristic function of the process, for a finite interval of time T, can be readily derived by taking advantage of the independence of the pulses and performing a weighted sum over a random walk for each fixed K [9, 10]. Thus we write

$$\begin{aligned} F_{2V}(\xi_1, \xi_2; t)_T &= \sum_{K=0}^{\infty} \frac{(nT)^K}{K!} e^{-nT} [F_{2v}(\xi_1, \xi_2; t)_T]^K, \\ &= \exp\{nT[F_{2v}(\xi_1, \xi_2; t)_T - 1]\}, \end{aligned} \tag{6}$$

where n is the average number of pulses occurring per unit time. F_{2v} is the characteristic function for the variable v, representing an individual pulse, and F_{2V} is the characteristic function for the sum, V.

The finiteness of the time interval can be eliminated by first expressing the exponent as an average,

$$nT[F_{2v}(\xi_1, \xi_2; t)_T - 1] = nT\langle \exp(i\xi_1 v_1 + i\xi_2 v_2) - 1 \rangle, \tag{7}$$

in which the average must be carried out over all the random variables of a typical pulse, e.g. amplitude, shape, phase, duration, or time of occurrence. Let us maintain the average sign as a reminder of the presence of other random variables and explicitly consider the averages for the times of occurrence. If the time variables are transformed from t_1', t_2' to $t - t_1'$, $t_2' - t_1'$, then, taking advantage of the uniform distribution of the occurrence time, we have

$$W(t_1', t_2')\,dt_1'\,dt_2' = W(t_2' - t_1') \frac{1}{T} \tau\, d\!\left(\frac{t - t_1'}{\tau}\right) d(t_2' - t_1'), \tag{8}$$

where τ is the mean duration of a pulse, so that letting T increase indefinitely, we get

$$F_{2V}(\xi_1, \xi_2; t) = \exp\left\{\gamma \int_{-\infty}^{\infty} \langle\{\exp[i\xi_1 v_1(t_0) + i\xi_2 v_2(t_0 + t)]\} - 1\rangle \, d\left(\frac{t_0}{\tau}\right)\right\}, \tag{9}$$

where $\gamma = n\tau$ is the average noise "density," i.e. the average number of pulses per second multiplied by the average duration of a pulse.

The semi-invariants of V are readily obtained by expanding the inner exponential of Eq. (9), yielding

$$\lambda_{mn}(t) = \gamma\langle v(t_0)^m v(t_0 + t)^n\rangle. \tag{10}$$

One is at once tempted to say, from Eq. (10), that the semi-invariants of V are proportional to the moments of v. This is not entirely proper, because the presence of the -1 under the average in Eq. (9) acts as a convergence factor, so that the exponent is not a true characteristic function, and hence the right hand side of Eq. (10) is not a true moment. For a finite time interval, no convergence problems arise, and Eq. (6) shows that then indeed the semi-invariants of V are proportional to the moments of v.

The parameter of the Poisson ensemble which has most influence in determining the general features of the noise is the density γ. As the noise density increases, the noise distributions tend toward the gaussian. When the density is small, the noise pulses overlap only slightly so that the noise has the attributes of a deterministic, interfering signal, whose time-structure is essentially that of a single typical impulse. These two limiting regions are important, since the exact distributions are usually too complex to be used analytically, and must be approximated differently in each of the two cases.

2.3 Narrow band noise. Here, in the course of rectification, the detector or rectifier is preceded by a frequency selective network which passes only a band of frequencies narrow compared to the center frequency of the band. This is simultaneously desirable in order to discriminate against unwanted signals in other frequency bands, and more or less inescapable because of the inherent characteristics of the elements of which the receiver is built. Thus, although the noise at the input of the receiver may be, and usually is, of relatively constant strength over a wide band of frequencies, the noise presented to the detector is narrow-band because of its passage through the frequency selective parts of the receiver.

Important simplifications of the probability densities can be obtained by taking explicit cognizance of the narrow-band nature of the processes with which we are concerned. A narrow-band random variable can be expressed as

$$V(t) = R(t) \cos[\omega_0 t - \theta(t)], \tag{11}$$

where ω_0 is the central frequency of the noise spectrum and R and θ are random variables whose variation with time is slow compared to that of $\cos \omega_0 t$. Thus, if one calculates the moments of V using time averages (over a single member function of the V ensemble, which is assumed to be ergodic), R and θ can be assumed to be constant over a single period of $\cos \omega_0 t$. Consequently, all the moments of V which are of odd degree vanish. The constancy of envelope and phase over one period of the center frequency is, of course, an approximation and cannot be expected to be valid for moments of arbitrarily high degree, since a change in R of δR will give a change in R^n of $nR^{n-1}\,\delta R$, and thus, for sufficiently large n, will be comparable to the change in $\cos \omega_0 t$. In the approximating distributions which will be used, however, only low-degree moments will appear, so that the invariance of the slowly varying parts over a single period of $\cos \omega_0 t$ is a valid assumption.

The presence of high frequency terms, such as $\cos \omega_0 t$, in the expression for the second-order moments is not eliminated, by any means. In order to indicate clearly the presence of high frequency terms, let us define an envelope factor for the moments, by

$$M_{mn}(t) \equiv \frac{\Gamma([m + n + 1]/2)}{\Gamma(1/2)\Gamma([m + n]/2 + 1)} \langle R_1^m R_2^m \rangle. \tag{12}$$

Let us further assume that the phase change of the noise voltage, i.e. $\theta_2 - \theta_1$, is not a random variable*. This is not a valid restriction on some types of random waves; but, in this paper, we shall need the results below in their application to individual pulses [the v_i of Eq. (5)], where the phase would be expected to be constant.

The second-order moments can now be written

$$\begin{aligned} \mu_{11}(t) &= \langle R_1 \cos (\omega_0 t_0 - \theta_1) R_2 \cos (\omega_0 t_0 + \omega_0 t - [\theta_1 + \alpha]) \rangle \\ &= \langle R_1 R_2 \cos^2 (\omega_0 t_0 - \theta_1) \rangle \cos (\omega_0 t - \alpha) \\ &\quad - \langle R_1 R_2 \cos (\omega_0 t_0 - \theta_1) \sin (\omega_0 t_0 - \theta_1) \rangle \sin (\omega_0 t - \alpha) \\ &= \tfrac{1}{2} \langle R_1 R_2 \rangle \cos (\omega_0 t - \alpha) \\ &= M_{11}(t) \cos (\omega_0 t - \alpha), \end{aligned} \tag{13a}$$

and similarly

$$\mu_{31}(t) = M_{31}(t) \cos (\omega_0 t - \alpha), \tag{13b}$$

$$\mu_{22}(t) = M_{22}(t) [\tfrac{1}{3} + \tfrac{2}{3} \cos^2 (\omega_0 t - \alpha)], \tag{13c}$$

which, with $\mu_{m,n}(0) = \mu_{m+n,0}$, defines all the non-zero moments up to the sixth degree.

We note that by comparing the two expansions of the characteristic function, Eqs. (3) and (4), a similar separation of high and low frequency factors can be accomplished for the semi-invariants, e.g.

$$\lambda_{11}(t) = \Lambda_{11}(t) \cos (\omega_0 t - \alpha), \tag{14}$$

and that a special notation is usually employed for the autovariance function, i.e.

$$\mu_{11}(t) = R(t) = \psi r(t) = \psi r_0(t) \cos (\omega_0 t - \alpha), \tag{15}$$

where $r(0) = r_0(0) = 1$.

2.4 Nearly gaussian distributions. A nearly normal distribution, e.g. Eq. (9) when γ is large, can be expanded asymptotically in terms of the limiting gaussian form. For the case where the odd degree semi-invariants are zero, the leading terms of the expansion of the characteristic function are

$$\begin{aligned} F_2(\xi_1, \xi_2; t) = \left\{ 1 + \frac{\lambda_{40}}{4!} \xi_1^4 + \frac{\lambda_{31}(t)}{3!} \xi_1^3 \xi_2 + \frac{\lambda_{22}(t)}{2!\,2!} \xi_1^2 \xi_2^2 + \frac{\lambda_{13}(t)}{3!} \xi_1 \xi_2^3 + \frac{\lambda_{04}}{4!} \xi_2^4 + \cdots \right\} \\ \cdot \exp \left\{ -\frac{\psi}{2} [\xi_1^2 + 2\xi_1 \xi_2 r(t) + \xi_2^2] \right\}. \end{aligned} \tag{16}$$

In taking the Fourier transform of this equation, we note that the multiplications by ξ_i will become differentiations with respect to V_i, and the two-dimensional form of the well-known Edgeworth series is obtained. The terms containing the fourth degree semi-

*Formulae for the case where the phase change is random may be found in Mullen and Middleton [11].

invariants can be shown to be of order $\Lambda_{40}/\psi^2 \sim \gamma^{-1}$ and the neglected terms to be of order γ^{-2}.

An alternative form, whose derivation is given in Appendix A, can be obtained in which derivatives are taken with respect to the second moments,

$$F_2(\xi_1, \xi_2; t) = [1 + L + 0(\gamma^{-2})] \cdot \exp\{-\tfrac{1}{2}[\psi_1\xi_1^2 + 2\xi_1\xi_2\psi r_0(t)\cos(\omega_0 t - \alpha) + \psi_2\xi_2^2]\}\,|_{\psi_1=\psi_2=\psi},$$

where

$$L = \frac{\Lambda_{40}}{3!}\frac{\partial^2}{\partial\psi_1^2} + \frac{2}{3!}\Lambda_{31}(t)\frac{\partial^2}{\partial\psi_1\,\partial\psi r_0} + \frac{\Lambda_{22}(t)}{3!}\left[\frac{\partial^2}{\partial\psi r_0^2} + 2\frac{\partial^2}{\partial\psi_1\psi_2}\right] + \frac{2\Lambda_{13}(t)}{3!}\frac{\partial}{\partial\psi_2\psi r_0} + \frac{\Lambda_{04}}{3!}\frac{\partial^2}{\partial\psi_2^2}, \tag{17}$$

and subscripts have been placed on the second moments to enable each term in the exponent to be differentiated separately.

Only semi-invariants of low degree appear, so that the narrow-band approximation is still valid. Equation (17) is applicable to any narrow-band, nearly normal distribution, whether derived from the Poisson ensemble or not.

For moments after a non-linear operation, the comparative simplicity in the analysis when the noise possesses a gaussian distribution is a strong point in favor of the series in parametric derivatives. Applying the differential operator to the result for (nonstationary) gaussian noise is likely to be involved, but is certainly straightforward. Furthermore, where the narrow-band structure of the output is important, this characteristic function possesses the additional advantage of presenting the high-frequency part only once (in the exponent) while the Edgeworth form has high-frequency terms of different orders scattered about among the various semi-invariants, as well.

2.5 Low-density distributions. When the density is small, the exact form of the distribution is too difficult to use, and at the same time, the Edgeworth type of approximation is no longer applicable. However, a useful ascending power series in γ can be obtained, starting from Eq. (9). Let

$$f(\xi_1, \xi_2; t) = 1 + \int_{-\infty}^{\infty} \langle \exp(i\xi_1 v_1 + i\xi_2 v_2) - 1\rangle\, d\left(\frac{t}{\tau}\right), \tag{18}$$

so that

$$F_{2V}(\xi_1, \xi_2; t) = \exp\{\gamma[f(\xi_1, \xi_2; t) - 1]\} = e^{-\gamma}\sum_{n=0}^{\infty}\frac{\gamma^n}{n!} f(\xi_1, \xi_2; t)^n, \tag{19}$$

a form which much resembles Eq. (6).

The utility of Eq. (19) lies in the fact that the first two terms can be used in their exact form and the remaining terms are similar to nearly gaussian distributions, where n corresponds to the noise density. After considerable algebra, one obtains

$$\begin{aligned} F_{2V}(\xi_1, \xi_2; t) = e^{-\gamma} + \gamma e^{-\gamma}\cdot f(\xi_1, \xi_2; t) + \sum_{n=2}^{\infty}\frac{\gamma^n}{n!}e^{-\gamma}\Bigg\{\left[1 - \frac{\psi^2}{2n}\cdot\frac{\partial^2}{\partial\psi^2}\right] \\ \cdot\exp\left\{-\frac{n\psi}{2\gamma}[\xi_1^2 + 2\xi_1\xi_2 r_0(t)\cos(\omega_0 t - \alpha) + \xi_2^2]\right\} + \frac{\gamma L}{n} \\ \cdot\exp\left\{-\frac{n}{2\gamma}[\psi_1\xi_1^2 + 2\psi\xi_1\xi_2 r_0(t)\cos(\omega_0 t - \alpha) + \psi_2\xi_2^2]\right\}\Bigg|_{\psi=\psi_1=\psi_2}\Bigg\}, \end{aligned} \tag{20}$$

where L is the differential operator that appears in Eq. (17). Since our aim is to obtain the covariance function after rectification, which in our representation of the rectifier (*vide* Eq. (22) *infra*), is itself a moment of the input distribution, although in general not one of integral degree, an Edgeworth series in which the low-degree moments are reproduced exactly is appropriate. As a fit to the distribution itself, however, the approximation cannot be expected to be nearly so good.

2.6 Example. The first-order probability distribution of rectangular *c-w* pulses will serve to illustrate the behavior of the distributions of the process as the density is varied.

The distributions corresponding to the terms of fixed n in Eq. (19) have been previously calculated [12]*. From these, one can easily obtain results, which are shown in Fig. 1, for the Poisson ensemble together with a nearly gaussian approximation for $\gamma = 1$ and a gaussian distribution. The figure is a plot of the probability that the value of the abscissa, measured in units of the standard deviation, will be exceeded in absolute value.

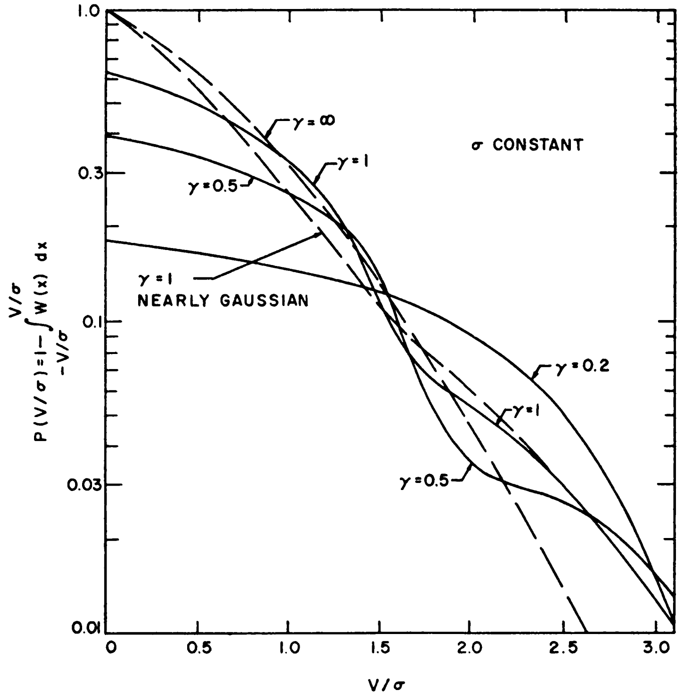

FIG. 1. Non-gaussian probability distributions.

When the noise density is one, the nearly gaussian approximation is a fairly close fit to the tail of the distribution, but is significantly different for small values of V. With the Poisson ensemble there is a finite probability of having no noise at all, so that

*Just as in Eq. (10) the right-hand side is not properly a moment, so in Eq. (19), f is not properly a characteristic function. However, the appearance of the anomalous behavior is entirely in the time structure of the process so that, for the first-order distribution, each term of the power series in γ can be considered as a characteristic function.

the non-gaussian distribution should contain a δ-function of strength $e^{-\gamma}$ at the origin. The nearly gaussian expansion, because of its asymptotic character, is unable to represent discrete probabilities.*

When plotted against V, the low-density curves lie above one another; but dividing this abscissa by the standard deviation, which varies as $\gamma^{1/2}$, causes the curves to intersect. As the density decreases, the probability that the noise will be zero becomes larger. This in turn decreases the standard deviation faster than it does the tail of the distribution, so that the probability of relatively high values increases, for smaller noise densities, relative to the larger values of noise density.

Section III. Properties of the rectifier output. 3.1 Introduction. Our next task is to use the distributions found in Sec. II to determine the autovariance function of the output of an *a-m* receiver when an additive mixture of *c-w* signal and non-gaussian noise is impressed on the input, viz.;

$$V_{IN}(t) = V_N(t) + A_0 \cos \omega_0 t. \tag{21}$$

The receiver in an amplitude modulation system contains a band-pass filter which eliminates all spectral components except those in a narrow-band around the central frequency to which it is tuned, and a demodulator, consisting of a half-wave νth-law detector followed by a low-pass filter. The relation between output and input of a non-linear device is the dynamic transfer characteristic. For the half-wave (zero memory) νth-law device, it is

$$I = g(V) = \begin{cases} \beta V^\nu, & V > 0 \\ 0, & V < 0 \end{cases}, \tag{22}$$

where I is the instantaneous output, V the instantaneous input, and β is an appropriate proportionality constant. The non-linear device produces an output whose spectral components lie in zones around multiples of the central frequency of the input [14]; the (ideal) low-pass filter eliminates all these zones except the low-frequency one corresponding to the zeroth harmonic.

We wish now to find the autovariance function (or since the input process is assumed to be ergodic, the correlation function) of the output. This is by no means the most complete statistical description that one could wish for, although the correlation function yields considerable insight into the nature of the output. However, to obtain the probability distributions of the low-frequency zone is a very difficult problem, which has been solved only for the quadratic detector with gaussian input noise (since no moments higher than the fourth order are then required) [15, 16].

Some general statements about the output covariance function can be made immediately. The covariance function equals the mean square of the output when its argument is zero. As the argument, t, increases beyond bounds, the two functions to be averaged become uncorrelated, so that the covariance function becomes the square of the output mean. The behavior peculiar to $R(t)$ is revealed by its variations with time, which can best be separated by defining a normalized output covariance function

$$r_{\text{out}}(t) \equiv \frac{R(t) - R(\infty)}{R(0) - R(\infty)}. \tag{23}$$

*A similar situation arises in the discussion of the Brownian motion, where a δ-function appears in the exact form of the velocity distribution which the usual approach of Fokker and Planck is unable to provide [13].

3.2 Noise models. To obtain the output covariance as a specific function of time, a definite time dependence must be assigned to the semi-invariants of the noise. An immense variety of types is naturally possible; three models felt to be important are treated here. In these models the entire time dependence is contained in the envelope of the pulses, although in general, variations in amplitude, shape, and phase of the individual pulses will also affect the time dependence. Variations in the amplitude or shape appear in the semi-invariants in the same way as the variations of pulse envelope, so that no loss in generality is to be expected if we attribute the overall time variation to a single cause rather than to a mixture of several. This is not true for the phase variation, however, since phase changes are associated with the high-frequency part of the semi-invariants, and therefore affect the noise distributions in a manner different from the amplitudes or pulse envelopes. Most noise sources produce pulses of constant phase, so that further refinements are unnecessary; however, the commonest exception, moving clutter in a radar system, is certainly an important one.*

Pertinent data on the three models are summarized in Table 1. The exponential pulses represent impulses passed through a single-tuned circuit. Although a single selective element is not an accurate model of the tuned stages of a receiver, this type of time dependence is important because it is necessary (and sufficient) if the process is to be Markoffian in the limit of increasing density. A representative physical case would be that of atmospheric static interference in a crystal video receiver.

The pass-band of an actual receiver is an involved function of the number of stages and the coupling networks between stages. As an abstraction from the details associated with any particular *i-f* strip, we shall take a pass-band of gaussian frequency response which preserves the essential features of a pass-band while remaining analytically tractable. Admittedly, a gaussian pass-band is not physically realizable, but it is a good approximation to the magnitude-frequency curve of actual amplifiers (if not to the phase frequency curve), and possesses the important virtue of simplicity**. Impulses passing through this *i-f* amplifier will become gaussian pulses, as in the second model of Table 1. "Gaussian" here refers to the pulse shape as a function of time and not to the statistics of the pulses.

The third type chosen is that of a square pulse envelope of finite duration. This exemplifies noise whose values are independent when separated by a sufficiently long, but finite time. The model fits the sonar and radar clutter problem when relative motion between the transceiver and scatterers is slight.

Notice that, in all these cases, the semi-invariants can be expressed as functions, in fact powers, of the input covariance function†. Accordingly, the time enters the output covariance only implicitly through the input covariance function.

3.3 Derivation of the output correlation function. The total covariance function of the output is the ensemble average of the product of the outputs as two times separated

*The alterations in the form of the covariance function which are necessary to take account of phase variations in time can be found in [11].

**H. Wallman has shown that the pass-band of cascaded networks whose individual step function responses have no overshoot tends toward the gaussian [17]. Furthermore, it seems likely that one can extend these results to any cascaded network except those tuned for a Butterworth response.

†The linear model is one extreme of possible time behavior in that its semi-invariants of all orders decrease no faster than the correlation function. One may conjecture that no $\Lambda_{22}(t)$, see Table 1, can decrease faster than the square of the covariance function, in which case the two other models represent the opposite extreme; however, we have been unable to prove this.

TABLE 1

Noise Models.

Pulse envelope $h(x)$	$\begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$	e^{-x^2}	$\begin{cases} 1, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$
Normalized covariance function $r_0(t)$	$e^{-\beta\|t\|}$	$e^{-\beta^2 t^2/2}$	$\begin{cases} 1 - \beta\|t\|, & \beta\|t\| < 1 \\ 0, & \beta\|t\| > 1 \end{cases}$
Normalized semi-invariants			
$\dfrac{\Lambda_{40}}{3!\psi^2}$	$1/4\gamma$	$1/4\gamma(\pi)^{\frac{1}{2}}$	$1/4\gamma$
$\dfrac{\Lambda_{31}(t) + \Lambda_{13}(t)}{3!\psi^2}$	$\dfrac{r_0 + r_0^3}{4\gamma}$	$r_0^{3/2}/2\gamma(\pi)^{\frac{1}{2}}$	$r_0/2\gamma$
$\dfrac{\Lambda_{22}(t)}{3!\psi^2}$	$r_0^2/4\gamma$	$r_0^2/4\gamma(\pi)^{\frac{1}{2}}$	$r_0/4\gamma$
$\dfrac{2^{(m+n)/2}}{(m+n)!}\dfrac{\Lambda_{mn}(t)}{\psi^{(m+n)/2}}$	$\dfrac{\gamma(2/\gamma)^{(m+n)/2}}{(m+n)\Gamma^2\left(\dfrac{m+n}{2}+1\right)}\begin{cases} r_0^m, & t > 0 \\ r_0^n, & t < 0 \end{cases}$	$\dfrac{\gamma(\pi)^{\frac{1}{2}}\left[\dfrac{1}{\gamma}\left(\dfrac{2}{\pi}\right)^{\frac{1}{2}}\right]^{(m+n)/2} r_0^{2mn/(m+n)}}{(m+n)^{\frac{1}{2}}\Gamma^2\left(\dfrac{m+n}{2}+1\right)}$	$\dfrac{r_0}{\gamma^{[(m+n)/2]-1}\Gamma^2\left(\dfrac{m+n}{2}+1\right)}$

by an interval t. The average can be expressed as a suitable integral over the characteristic function. We have [14, 18]

$$R_T(t) = \langle g(V_1)g(V_2)\rangle = \frac{1}{4\pi^2}\iint_C d\xi_1\, d\xi_2 f(i\xi_1)f(i\xi_2)\langle \exp(iV_1\xi_1 + iV_2\xi_2)\rangle, \tag{24}$$

where the average defines the characteristic function and

$$f(i\xi) = \int_{-\infty}^{\infty} g(V)e^{-iV\xi}\, dV, \qquad \operatorname{Im}\xi < 0 \tag{25}$$

in which $g(V)$ is zero for V less than some V_0, and of no greater than exponential order at infinity, and **C** is a straight line in the complex ξ-plane parallel to the real axis and lying below the singularities of $f(i\xi)$.

Since the input signal ensemble is the sum (see (21)) of sine wave and noise ensembles, which are independent of each other, the characteristic function of Eq. (24) becomes the product of the separate characteristic functions of the signal and of the noise. The results of Sec. II enable us to obtain the covariance function for non-gaussian noise, if we can obtain the corresponding one for non-stationary gaussian noise. Since the non-stationary gaussian case is only a very slight extension of previous work, it can easily be solved by methods briefly described below.

The characteristic function of the sine wave signal $A_0 \cos(\omega_0 t + \varphi)$ is [14]

$$F_2(\xi_1\,,\xi_2\,;t)_S = J_0(A_0[\xi_1^2 + \xi_2^2 + 2\xi_1\xi_2 \cos\omega_0 t]^{1/2}) \tag{26}$$

and the characteristic function of the noise is given by the exponential of Eq. (17). In order to find the zonal structure of the output, the two characteristic functions can be expanded in Fourier series, giving for the characteristic function of Eq. (24)

$$F_2(\xi_1\,,\xi_2\,;t) = \exp\left(-\frac{\psi_1\xi_1^2 + \psi_2\xi_2^2}{2}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{2}\epsilon_m\epsilon_n J_n(A_0\xi_1)J_n(A_0\xi_2)$$
$$[I_{m+n}(\psi r_0\xi_1\xi_2)\cos(m\omega_0 t + n\alpha) + I_{|m-n|}(\psi r_0\xi_1\xi_2)\cos(m\omega_0 t - n\alpha)], \tag{27}$$

where the Neumann factor ϵ_m equals one for $m = 0$ and is two for all other values.

For the output of the receiver, only the low-frequency zone, $m = 0$, is required*. If now the modified Bessel function is expanded in a power series, the double integral of Eq. (24) separates into two single integrals, which have been evaluated previously [19], so that the covariance function of the low-frequency zone is

$$R_G(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\epsilon_n(\psi^2r_0^2/\psi_1\psi_2)^{k+n/2}}{k!\,(k+n)!\,n!^2}\left(\frac{\psi_1\psi_2}{4}\right)^{\nu/2}(p_1p_2)^{n/2}\left(-\frac{\nu}{2}\right)^2_{n+k}\cos n\alpha$$
$$\cdot {}_1F_1(-\nu/2+n+k;n+1;-p_1)\,{}_1F_1(-\nu/2+n+k;n+1;-p_2), \tag{28}$$

where p_i is the signal-to-noise power ratio, $A_0/2\psi_i$, ${}_1F_1$ is Kummer's form of the confluent hypergeometric function, and $(a)_n = \Gamma(a+n)/\Gamma(a)$.

3.4 Strong signals. When the signal is strong, i.e. p_i is large, Eq. (28) can be simplified by using the asymptotic expansion of the confluent hypergeometric function,

*Note that, if the ordinary asymptotic form of the characteristic function of the noise is used, all the harmonics must be retained until the cross-terms between the frequency-dependent semi-invariants have been taken into account.

which gives

$$R_G(t) \simeq \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^4(\nu/2+1)}\left(\frac{A_0^2}{4}\right)^\nu \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\epsilon_n}{k!\,(k+n)!}(-\nu/2)_k^2(-\nu/2)_{n+k}^2(r_0/p)^{n+2k}\cos n\alpha$$

$$\cdot {}_2F_0(-\nu/2+n+k,\ -\nu/2+k;\ 1/p_1)\cdot {}_2F_0(-\nu/2+n+k,\ -\nu/2+k;\ 1/p_2), \qquad (29)$$

where the formal hypergeometric function ${}_2F_0$ is defined as

$${}_2F_0(a,\ b;\ x) = \sum_{n=0}^{\infty}(a)_n(b)_n\frac{x^n}{n!}. \qquad (30)$$

The asymptotic expansions used here are valid only when $p \gg |\,-\nu/2+n+k\,|$. However, it can be shown [20] that the remainder contributed by the various possible series, e.g. when $n \gg k \gg p$, $k \gg p \doteq n$, etc., is of higher order than the remainder indicated by taking a finite part of Eq. (31), provided that $p > |\,-\nu/2+n+k\,|$ for all terms which are included.

The differentiation of the gaussian result to obtain the non-gaussian one is readily accomplished with the help of the formula

$$\frac{d}{dx}\,{}_2F_0(a,\ b;\ cx) = abc\ {}_2F_0(a+1,\ b+1;\ cx). \qquad (31)$$

When the leading terms of the power series are collected together, we find finally that the high and low density cases give the same result, viz:

$$\begin{aligned} R(t) \simeq \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^4(\nu/2+1)}\left(p\,\frac{\psi}{2}\right)^\nu \Big\{1 &+ 2(\nu/2)^2\,\frac{1+r_0}{p} + \frac{(\nu/2)^2}{p^2}\,[(\nu/2)^2(1+r_0^2) \\ &+ (\nu/2-1)^2(1+r_0^2\cos 2\alpha) + 4(\nu/2)(\nu/2-1)r_0\cos\alpha] \\ &+ \frac{(\nu/2)^2}{3p^2\psi^2}\,[(\nu/2-1)^2\Lambda_{40} + 2\nu/2(\nu/2-1)[\Lambda_{31}(t)+\Lambda_{13}(t)] \\ &+ [2(\nu/2)^2 + (\nu/2-1)^2]\Lambda_{22}(t)] + 0(p^{-3}\gamma^{-2})\Big\}. \end{aligned} \qquad (32)$$

The order of the remainder is different as the noise density becomes large or small. The terms in p^{-2} are of order p^{-2} and $p^{-2}\gamma^{-1}$; the next set of terms are of order p^{-3}, $p^{-3}\gamma^{-1}$, and $p^{-3}\gamma^{-2}$. For nearly gaussian noise, γ is large so that Eq. (32) can be used if p alone is large. For low-density noise, however, γp must be large as well as p.

In physical terms, γp is the ratio of signal power to the noise power in a single noise pulse. When the noise density is small, there is no noise at all for an appreciable part of the time; thus the requirement on γp means that the signal must be relatively strong while the noise pulse is present, which is here a more stringent requirement than that it be strong on the average.

When the signal is strong, there is, as usual, a modulation suppression effect [14] in that the leading term is the one for signal alone, with noise entering as a first-order correction. Here, however, there is an additional suppression effect on the statistics of the noise. As Eq. (32) shows, the non-gaussian nature of the noise is first discernible in the second-order terms.

3.5 Weak signals. The results when the signal is weak are more complicated because the low-density and high-density cases can no longer be described by the same formula.

If the confluent hypergeometric functions of Eq. (20) are expressed in series, one sum can be obtained in closed form, leaving a triple power series in the p_i, viz;

$$R_G(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\left(\frac{\psi_1\psi_2}{4}\right)^{\nu/2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\epsilon_n(-1)^{k+l}}{k!\,l!\,n!}\frac{p_1^{k+n/2}}{(k+n)!} \cdot\frac{p_2^{l+n/2}}{(l+n)!}\left(\frac{\psi^2 r_0^{2n/2}}{\psi_1\psi_2}\right)\cos n\alpha\; \mathfrak{F}\left\{0,0,0;\frac{\psi^2 r_0^2}{\psi_1\psi_2}\right\}, \tag{33}$$

where

$$\mathfrak{F}\{a,b,c;x\} = (-\nu/2)_{k+n+a}(-\nu/2)_{l+n+a} \cdot {}_2F_1(-\nu/2+k+n+a,\,-\nu/2+l+n+b;n+1+c;x). \tag{34}$$

Applying the differential operator of Eq. (17) one finds for the covariance function for nearly gaussian noise

$$R(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\left(\frac{\psi}{2}\right)^{\nu} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\epsilon_n(-1)^{k+l}\cos n\alpha}{n!\,(n+k)!\,(n+l)!} \cdot\frac{r_0^n p^{n+k+l}}{k!\,l!}\left\{\mathfrak{F}\{0,0,0;r_0^2\} + \frac{1}{3\psi^2}G(t)\right\}, \tag{35}$$

where

$$\begin{aligned} G(t) = {}& \Lambda_{40}\mathfrak{F}\{2,0,0;r_0^2\} + [\Lambda_{31}(t)+\Lambda_{13}(t)]\left[\frac{n}{r_0}\mathfrak{F}\{1,0,0;r_0^2\}\right. \\ &\left. + \frac{2r_0}{n+1}\mathfrak{F}\{2,1,1;r_0^2\}\right] + \Lambda_{22}(t)\left[\frac{n(n-1)}{2r_0^2}\mathfrak{F}\{0,0,0;r_0^2\}\right. \\ &\left. + \frac{3n+2}{n+1}\mathfrak{F}\{1,1,0;r_0^2\} + \frac{r_0^2}{(n+1)^2(n+2)}\mathfrak{F}\{2,2,2;r_0^2\}\right]. \end{aligned} \tag{36}$$

The hypergeometric functions reduce to polynomials when ν is an even integer and to complete elliptic integrals when ν is an odd integer.

When the noise density is small, one must use the characteristic function Eq. (20), which involves much the same sort of derivative as does the high-density case just treated. In addition, however, there are the time intervals in which no noise pulse or one noise pulse occurs. If there is no noise present, the correlation function is that for signal alone, which was found as the first term of Eq. (32).

If a single noise pulse alone is present, the exact form of the distributions must be used. On the other hand, when the signal is present at the same time as a single noise pulse, the covariance function is less sensitive to the statistics of the noise, and it becomes possible to approximate the noise distribution in the usual way. For a single pulse the covariance function is most easily found by not carrying out the average until the end. Thus the characteristic function of the noise can be written as

$$F_1(\xi_1, \xi_2; t) = \langle J_0([v_1^2\xi_1^2 + v_2^2\xi_2^2 + 2\xi_1\xi_2 v_1 v_2\cos(\omega_0 t - \alpha)]^{1/2}\rangle, \tag{37}$$

where the average is over the parameters of $v(t_0 - \tau)$, an individual noise pulse.

This characteristic function can be expanded in a Fourier series by the usual Bessel function addition theorem, whereupon the double integral for the low frequency zone splits into the product of single integrals each of which is of Weber's type [21], so that the correlation function is

$$\frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^4(\nu/2+1)}\left(\frac{\psi}{2}\right)^\nu \frac{\langle v_1^\nu v_2^\nu\rangle}{\gamma^\nu\langle v^2\rangle}. \tag{38}$$

If now the differential operator of Eq. (20) is applied to Eq. (33), we obtain finally for a weak signal and low-density noise

$$\begin{aligned} R(t) = \frac{\beta^2\Gamma^2(\nu+1)}{4\Gamma^2(\nu/2+1)}\left(\frac{\psi}{2}\right)^\nu &\left\{\frac{p^\nu e^{-\gamma}}{\Gamma^2(\nu/2+1)}\right. \\ &+\frac{e^{-\gamma}}{\gamma^{\nu-1}\Gamma^2(\nu/2+1)}\frac{\langle v_1^\nu v_2^\nu\rangle}{\langle v^2\rangle^\nu} \\ &+\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{l=0}^\infty \frac{\epsilon_n(-1)^{k+l}r_0^n\cos n\alpha}{n!\,(n+k)!\,k!\,(n+l)!\,l!}p^{k+l+n} \\ &\cdot[\gamma^{-\nu+n+k+l}\varphi(\gamma,\nu-n-k-l)\mathfrak{F}\{0,0,0;r_0^2\} \\ &+\frac{1}{3\psi^2}\gamma^{1-\nu+n+k+l}\varphi(\gamma,\nu-1-n-k-l)G(t) \\ &-\tfrac{1}{2}(\nu-k-l-n)(\nu-k-l-k-1) \\ &\qquad\left.\cdot\gamma^{-\nu+n+k+l}\varphi(\gamma,\nu-n-k-l)\mathfrak{F}\{0,0,0;r_0^2\}]\right\}, \end{aligned} \tag{39}$$

where $G(t)$ is defined in Eq. (36) and

$$\phi(\gamma,\beta) \equiv \sum_{k=1}^\infty \frac{\gamma^k}{k!}k^\beta e^{-\gamma}; \tag{40}$$

φ is a polynomial in γ and $\exp(-\gamma)$ for positive integral β, and an iterated integral of the exponential integral for negative integral β; for other values, a closed form is unavailable.

3.6 The quadratic detector. The output correlation function after a half-wave quadratic detector is particularly simple. When ν equals 2, the output voltage is the same, except for a scale factor, as that from a full-wave squaring device, because the noise and signal distributions are symmetric about $v = 0$. The covariance function of all zones is thus a fourth-degree moment of the input. The covariance function of the low-frequency zone is then

$$R(t) = \frac{\beta^2\psi^2}{4}\left[1 + r_0^2 + \frac{2\Lambda_{22}(t)}{3\psi^2} + 2p(1+r_0) + p^2\right], \tag{41}$$

where $p = A_0^2/2\psi$, the input signal-to-noise ratio, and $r_0 = r_0(t)$. The equation is exact for all values of signal-to-noise ratio or noise density. Finding the output covariance function as a moment of the input voltage is by far the simplest approach; however, the method may be applied only when ν is an even integer.

Results for the three noise models listed above in Sec. 3.2 are given in Table 2 and illustrated for the exponential and linear models in Figs. 2, 3, and 4. The figures show that the square-law rectifier reduces the amount of correlation compared to that present

TABLE 2

Covariance functions for the quadratic detector.

1. Exponential model:

$$r_0 = e^{-\beta|t|}$$

$$R(t) = \frac{\beta^2\psi^2}{4}\left[(1+p)^2 + 2pr_0 + \left(1+\frac{1}{\gamma}\right)r_0^2\right]$$

2. Gaussian model: $$r_0 = \exp\left(-\frac{\beta^2t^2}{2}\right)$$

$$R(t) = \frac{\beta^2\psi^2}{4}\left[(1+p)^2 + 2pr_0 + \left(1+\frac{1}{\gamma(\pi)^{1/2}}\right)r_0^2\right]$$

3. Linear model: $$r_0 = \begin{cases} 1-\beta\,|\,t\,|, & \beta\,|\,t\,| < 1 \\ 0, & \beta\,|\,t\,| > 1 \end{cases}$$

$$R(t) = \frac{\beta^2\psi^2}{4}\left[(1+p)^2 + \left(2p+\frac{1}{\gamma}\right)r_0 + r_0^2\right]$$

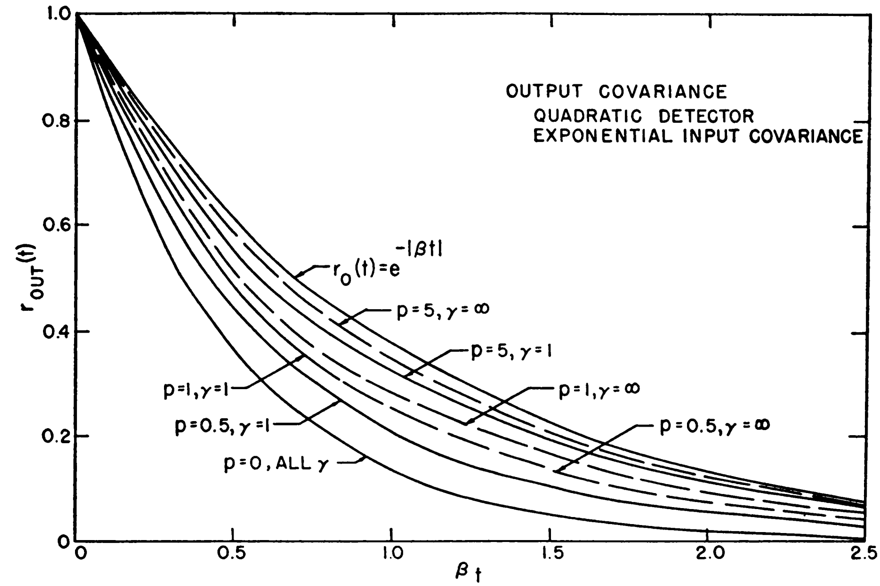

FIG. 2. Output covariance for the quadratic detector (exponential input covariance).

in the input for all values of p and γ. As p increases, the curves for all values of noise density increase too. This is because the signal × noise intermodulation terms become progressively more important as compared to the noise × noise products and, since the $s \times n$ contribution is more correlated*, $r_{out}(t)$ is increased. For large p a non-gaussian

*The expression "more correlated" is used throughout the paper to express the fact that one normalized output correlation function is greater than another for the same value of t when both are derived from the same input correlation function.

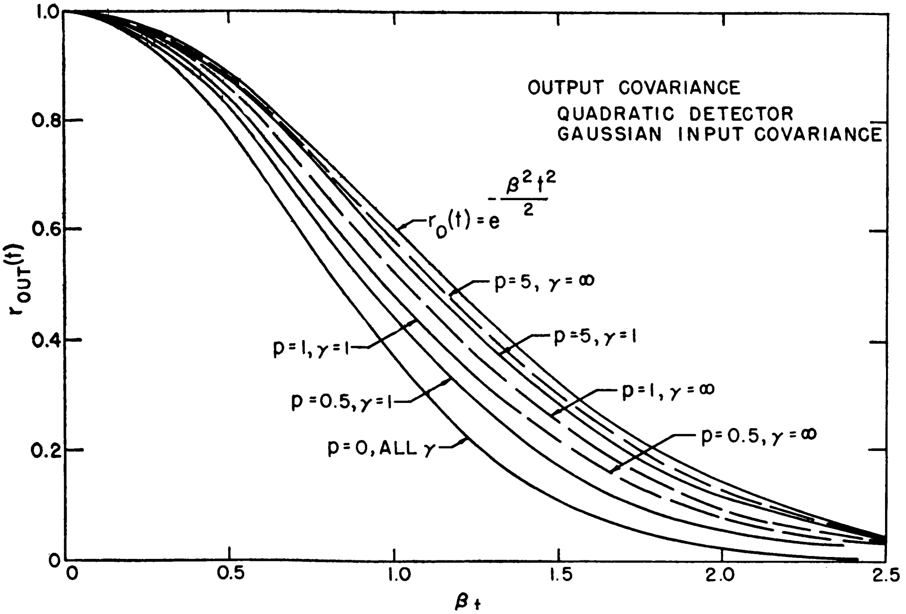

FIG. 3. Output covariance for the quadratic detector (gaussian input covariance).

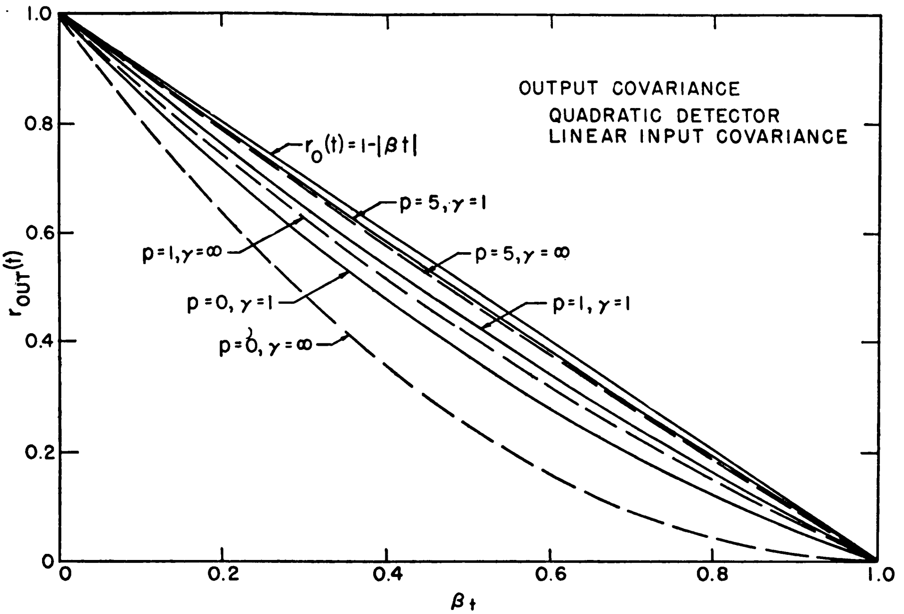

FIG. 4. Output covariance for the quadratic detector (linear input covariance).

suppression effect enters, so that the difference between gaussian and non-gaussian noise is less than at low signal levels.

The effect of the non-normal nature of the noise here is to reduce the output correlation compared to the corresponding gaussian noise for the exponential and gaussian noise models and to increase it for the linear model. Because of the simplicity of the expression for the covariance function, it is easy to see why this is so. For gaussian input noise, the quadratic detector output has one distorted noise term (r_0^2, for $n \times n$ modulation). For non-normal noise, an additional term enters, containing the semi-invariant $\Lambda_{22}(t)$. This term adds to the "scrambled" noise for the exponential and gaussian models, and to the undistorted noise for the linear model. Clearly, the output correlation is usually decreased by non-gaussian statistics since, for nearly all variations, $\Lambda_{22}(t)$ will decrease faster than $r_0(t)$. The linear model is the extreme case in which this is not so.

3.7 The linear detector. Obtaining the correlation function for a half-wave linear detector involves all the considerations, if not quite the full algebraic complexity, which attend the general case. No elementary method of obtaining the output covariance function is possible for a half-wave linear detector. However, when the general νth-law results of preceding sections are specialized to $\nu = 1$, considerable simplification occurs. The hypergeometric functions of the general result reduce to complete elliptic integrals and the results for large and small density become much alike. Table 3 lists the correlation functions for the three noise models. The three cases of a strong signal, a weak signal with high-density noise, and a weak signal with low-density noise are listed separately. The expressions in the table are the leading terms of series in the noise density γ and signal-to-noise ratio p. The nearly gaussian and low-density results hold over a considerable range of variation of γ, because the numerical coefficients of higher terms are small. The strong signal formulas of the table are reasonably accurate down to values of p of about 2. The weak signal correlation function, on the other hand, will hold only in a restricted range, for p no more than about 1/2, because only the first two terms of the ascending p series have been used.

TABLE 3

Covariance functions for the linear detector(*).

a. Strong signal, $p > 1$

1. Exponential model: $r_0 = \exp(-\beta \mid t \mid)$

$$R(t) = \frac{2\beta^2\psi p}{\pi^2}\left\{1 + \frac{1 + r_0}{2p} + \frac{1 - r_0}{8p^2}\left[1 - r_0 + \frac{1}{4\gamma}(1 - r_0 + 2r_0^2)\right]\right.$$
$$\left. + \frac{1 - r_0^2}{16p^3}\left[1 - r_0 + \frac{1}{4\gamma}(5 - 5r_0 + 2r_0^2)\right] + 0(p^{-4}, \gamma^{-2})\right\}$$

2. Gaussian model: $r_0 = e - \dfrac{\beta^2 t^2}{2}$

$$R(t) = \frac{2\beta^2\psi p}{\pi^2}\left\{1 + \frac{1 + r_0}{2p} + \frac{1}{8p^2}\left[(1 - r_0)^2 + \frac{1}{4\gamma(\pi)^{1/2}}(1 - 4r_0^{3/2} + 3r_0^2)\right]\right.$$
$$\left. + \frac{1}{16p^3}\left[(1 - r_0^2)(1 - r_0) + \frac{1 + r_0}{4\gamma(\pi)^{1/2}}(5 - 8r_0 - 4r_0^{3/2} + 7r_0^2)\right\} + 0(p^{-4}, \gamma^{-2})\right.$$

*Here $K(r_0)$ and $E(r_0)$ are complete elliptic integrals of the first and second kind, respectively, and $B(r_0) = E + (1 - r_0^2)K$, $r_0^4C(r_0) = (2 - r_0^2)K - 2E$, $r_0^2D(r_0) = K - E$. (See Jahnke and Emde, "Tables of Functions," Dover, New York, 1945, pp. 73-80.) Also $\phi(\gamma, -1)$ is obtained from Eq. (40).

3. Linear model:

$$r_0 = \begin{cases} 1 - \beta \mid t \mid, & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$$

$$R(t) = \frac{2\beta^2 \psi p}{\pi^2} \left\{ 1 + \frac{1 + r_0}{2p} + \frac{1 - r_0}{8p^2} \left[1 - r_0 + \frac{1}{4\gamma} \right] + \frac{1 - r_0}{16p^3} \left[1 - r_0^2 + \frac{1}{4\gamma}(5 - 3r_0) \right] + 0(p^{-4}, \gamma^{-2}) \right.$$

b. Weak signal, low-density noise, $p < 1, \gamma < 1$

1. Exponential model: $r_0 = \exp(-\beta \mid t \mid)$

$$R(t) = \frac{\beta^2 \psi}{\pi^2} \left\{ [E(r_0) + r_0^2 B(r_0)][1 - e^{-\gamma}] + \frac{4}{\pi} \gamma e^{-\gamma} r_0 - (1 - r_0^2) \frac{1 - (1 + \gamma)e^{-\gamma}}{8\gamma} [2E(r_0) - K(r_0)] + p \left[\frac{4}{\pi} e^{-\gamma} + \{1 - e^{-\gamma}\}\{E(r_0) + r_0 B(r_0)\} + \frac{\phi'(\gamma, -1)}{8} \{(1 - 2r_0^2)E(r_0) + r_0 B(r_0)\} \right] + 0(p^2) \right\}$$

2. Gaussian model: $r_0 = \exp\left(-\frac{\beta^2 t^2}{2}\right)$

$$R(t) = \frac{\beta^2 \psi}{\pi^2} \left\{ [E(r_0) + r_0^2 B(r_0)][1 - e^{-\gamma}] + \frac{4}{\pi} \gamma e^{-\gamma} r_0 + \frac{1 - (1 + \gamma)e^{-\gamma}}{8\gamma(\pi)^{1/2}} [4r_0^2 D(r_0) - (1 - r_0^2)K(r_0) - 4r_0^{5/2} D(r_0)] + p \left[\frac{4}{\pi} e^{-\gamma} + \{1 - e^{-\gamma}\}\{E(r_0) + r_0 B(r_0)\} + \frac{\phi(\gamma, -1)}{8(\pi)^{1/2}(1 - r_0^2)} \{(1 - 5r_0^2 + 4r_0^{5/2})E(r_0) + r_0(3 - 4r_0^{1/2} + r_0^2)B(r_0)\} \right] + 0(p^2) \right\}$$

3. Linear model:

$$r_0 = \begin{cases} 1 - \beta \mid t \mid & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$$

$$R(t) = \frac{\beta^2 \psi}{\pi^2} \left\{ [E(r_0) + r_0 B(r_0)][1 - e^{-\gamma}] + \frac{4}{\pi} \gamma e^{-\gamma} r_0 - (1 - r_0) \frac{1 - (1 + \gamma)e^{-\gamma}}{8\gamma} [K(r_0) - 2r_0 D(r_0)] + p \left[\frac{4}{\pi} e^{-\gamma} + \{1 - e^{-\gamma}\}\{E(r_0) + r_0 B(r_0)\} + (1 - r_0) \frac{\phi(\gamma, -1)}{8} \left\{ \frac{3B(r_0)}{1 + r_0} - D(r_0) \right\} \right] + 0(p^2) \right\} \phi(\gamma, -1) = e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!n}$$

c. Weak signal, nearly gaussian noise, $p < 1, \gamma > 1$

1. Exponential model: $r_0 = \exp(-\beta \mid t \mid)$

$$R(t) = \frac{\beta^2\psi}{\pi^2}\left\{E(r_0) + r_0^2 B(r_0) - \left(\frac{1 - r_0^2}{8\gamma}\right)[2E(r_0) - K(r_0)] + p\left[E(r_0) + r_0 B(r_0) + \frac{1}{8\gamma}\{(1 - 2r_0^2)E(r_0) + r_0 B(r_0)\}\right] + 0(p^2, \gamma^{-2})\right\}$$

2. Gaussian model: $r_0 = \exp\left(-\frac{\beta^2 t^2}{2}\right)$

$$R(t) = \frac{\beta^2\psi}{\pi^2}\left\{E(r_0) + r_0^2 B(r_0) + \frac{1}{8\gamma(\pi)^{1/2}}[4r_0^2 D(r_0) - (1 - r_0^2)K(r_0) - 4r_0^{5/2} D(r_0)] + p\left[E(r_0) + r_0 B(r_0) + \frac{1}{8\gamma(1 - r_0^2)(\pi)^{1/2}}\{(1 - 5r_0^2 + 4r_0^{5/2})E(r_0) + r_0(3 - 4r_0^{1/2} + r_0^2)B(r_0)\}\right] + 0(p^2, \gamma^{-2})\right\}$$

3. Linear model: $r_0 = \begin{cases} 1 - \beta \mid t \mid, & \beta \mid t \mid < 1 \\ 0, & \beta \mid t \mid > 1 \end{cases}$

$$R(t) = \frac{\beta^2\psi}{\pi^2}\left\{E(r_0) + r_0^2 B(r_0) - \frac{1 - r_0}{8\gamma}[K(r_0) - 2r_0 D(r_0)] + p\left[E(r_0) + r_0 B(r_0) + \frac{1 - r_0}{8\gamma}\left\{\frac{3}{1 + r_0}B(r_0) - D(r_0)\right\}\right] + 0(p^2, \gamma^{-2})\right\}$$

The normalized output correlation function is shown in Figs. 5, 6, and 7 with $r_{out}(t)$ for gaussian noise of the same input power shown in dashed lines. One may note, as for the quadratic detector, that increases in p cause the amount of correlation in the output to increase and cause the difference between gaussian and non-gaussian noise to decrease; however, the results have several features which are distinct from those for the quadratic detector. For example, as the noise density decreases, the leading term in the low-density series, $\langle v^r(t_0)v^r(t_0 + t)\rangle$, which for the linear detector is $r_0(t)$, becomes increasingly important, so that the output has more correlation than the output for a gaussian input noise for all noise models. This effect will occur as the noise density becomes sufficiently small for fixed signal power, even though the (average) signal-to-noise ratio is large. When γp becomes small, the weak signal series must be used.

Determination of the extent to which the nearly normal and low-density results together cover the range of γ is an important question which the covariance functions for the linear detector partially answer. Apparently the shape of the noise pulses has considerable effect. For gaussian pulses, the two forms of the correlation function are in close agreement (too close to be resolved in the figure); for the exponential and linear pulses, the agreement is not so complete. The figures tend to show that when the noise density is about 1, the two forms of the covariance function will be only roughly equal, but that no gross differences occur.

3.8 Effects of non-gaussian statistics in general. The linear and quadratic detectors are the most important, so that the results applicable to them have been presented in

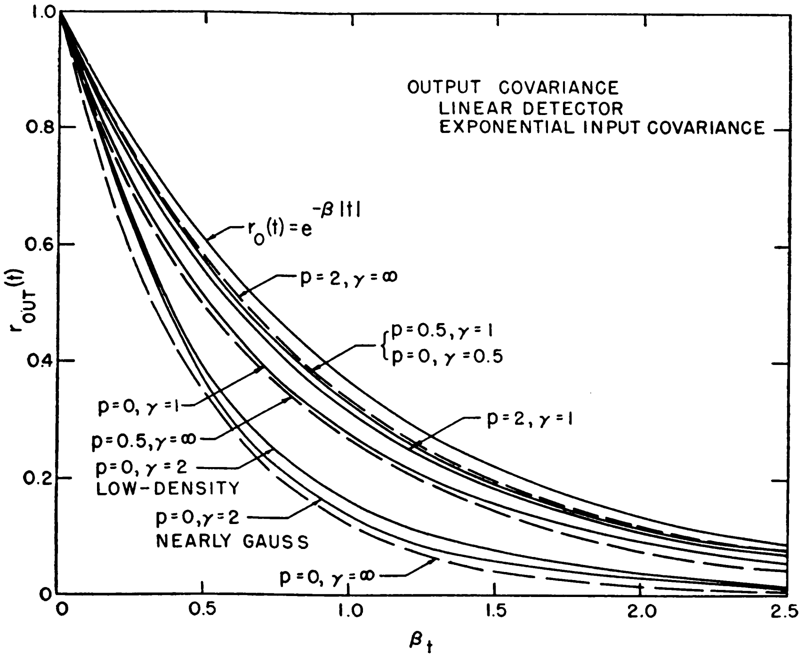

FIG. 5. Output covariance for the linear detector (exponential input covariance).

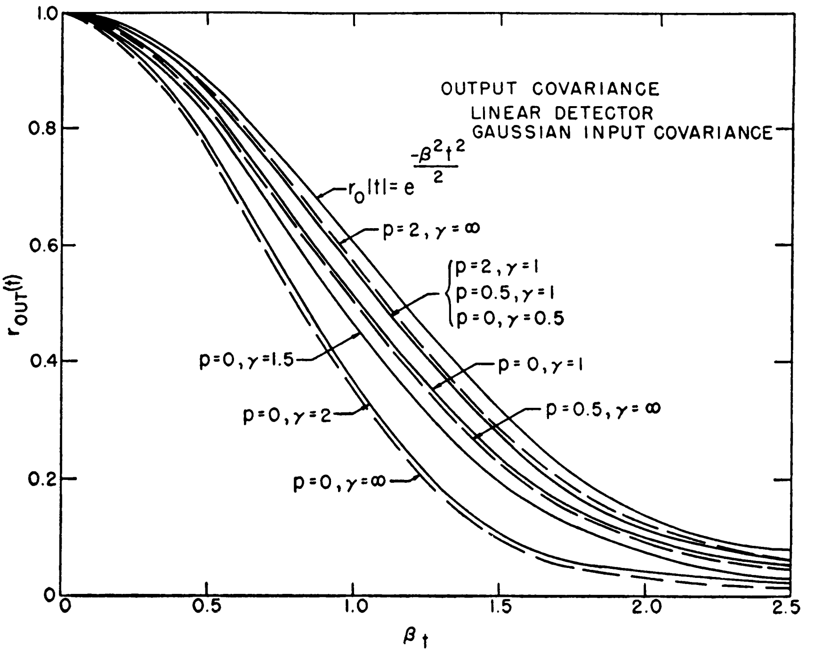

FIG. 6. Output covariance for the linear detector (gaussian input covariance).

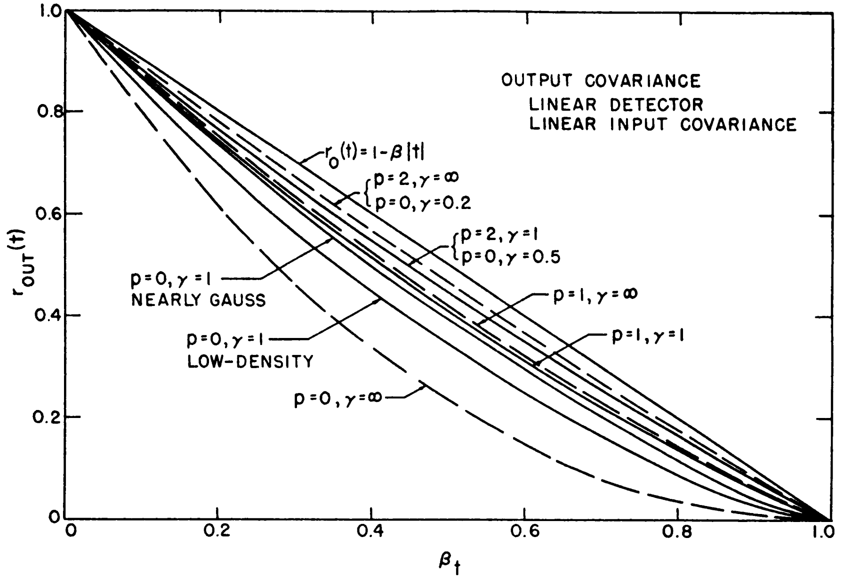

FIG. 7. Output covariance for the linear detector (linear input covariance).

considerable detail. A qualitative look at the output correlation function for arbitrary ν will indicate the over-all features of these results and indicate the place of the linear and quadratic detectors in the general scheme.

Comparison of the covariance functions of gaussian and non-gaussian noise can be carried out for the weak signal cases by comparing the leading terms of the expansions of the hypergeometric functions. Separate expansions must be used when r_0^2 is near zero and when r_0 is near one. Since in both limiting regions, the same results emerge from the analysis, we may expect with a reasonable degree of confidence that the results hold for all values of $r_0^2(t)$, and hence all t. If the signal is strong, all the functions involved are elementary, so that the comparison is much easier to make. In both cases, the algebra is at once straightforward, lengthy, and not especially illuminating, so that it will not be reproduced here [11, 22]. Figure 8 shows the results of such a comparison. In the shaded region the non-gaussian and gaussian correlation functions intersect each other, so that neither can be said to be larger, although for most values of t the non-gaussian output is less correlated. No scale has been indicated on the vertical axis because the position of the boundaries depends on the density of the noise. Deviations from the gaussian result are small for large signals and large for small signals.

As mentioned in Sec. 3.2, the noise models chosen here represent nearly the extremes of variation, so that we may expect the same general type of behavior for other shapes of noise pulses. Unless the semi-invariants of high degree have the same time structure [e.g. in the linear model, they all equal $r_0(t)$], the type of behavior shown by the gaussian or exponential noise model should be representative.

3.9 The output spectrum. The variation of the output power as a function of frequency is an important statistic of the output. It can be obtained as the cosine trans-

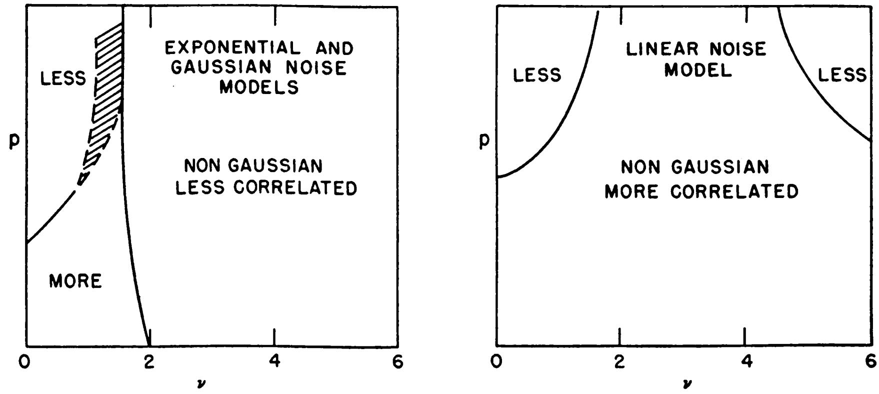

FIG. 8. General effect of non-gaussian statistics.

form of the covariance function,

$$W(f) = 4 \int_0^\infty R(t) \cos \omega t \, dt$$

$$R(t) = \int_0^\infty W(f) \cos \omega t \, df. \tag{42}$$

The covariance function is the quantity found directly from the analysis of the detector and therefore has been presented in greater detail than the spectrum will be. In general, the only tractable expressions for the covariance function are approximate, and the transform even of these can be taken only approximately. The covariance function and spectrum are equivalent in principle; and the propagation of error in taking cosine transforms can be minimized by giving results primarily in terms of the covariance function.

The general behavior of the spectrum can be inferred from the reciprocal spreading property of Fourier transforms, i.e. the more a function is concentrated in one domain, the more it is dispersed in the transform domain [23].

A more quantitative measure of the spectrum can be found by the use of equivalent rectangular bandwidths. The low-frequency output spectrum has a maximum at zero frequency and decreases smoothly as the frequency increases. One can construct a spectrum having a constant value over a finite bandwidth equal to the zero frequency value of the actual spectrum and the value zero elsewhere, such that the total power in the actual and rectangular spectra is the same for both. Then the total bandwidth of the rectangular spectrum is the equivalent rectangular bandwidth of the actual spectrum. It is customary not to include the δ-function at zero frequency representing d-c power in the actual spectrum.

Explicitly, one has

$$f_0 = \frac{\int_0^\infty W_0(f) \, df}{W_0(0)}, \tag{43}$$

where $W_0(f)$ is the spectrum minus the d-c contribution. In terms of the correlation function, the bandwidth is

$$f_0 = \frac{R(0) - R(\infty)}{4 \int_0^\infty [R(t) - R(\infty)]\, dt} = \left[4 \int_0^\infty r_{\text{out}}(t)\, dt \right]^{-1}. \tag{44}$$

Table 4 gives the ratio of the output bandwidth to the input bandwidth for all the curves of Figs. 2 to 7. The entries in the table give a quantitative measure of the difference

TABLE 4

*Spectral bandwidths**.

Quadratic Detector

Exponential covariance function			Gaussian covariance function			Linear covariance function		
$p = 0$,	all γ	2.00	$p = 0$,	all γ	1.41	$p = 0$,	$\gamma = \infty$	1.50
$p = .5$,	$\gamma = \infty$	1.33	$p = .5$,	$\gamma = \infty$	1.17	$p = 0$,	$\gamma = 1$	1.20
$p = .5$,	$\gamma = 1$	1.50	$p = .5$,	$\gamma = 1$	1.28	$p = 1$,	$\gamma = \infty$	1.12
$p = 1$,	$\gamma = \infty$	1.20	$p = 1$,	$\gamma = \infty$	1.11	$p = 1$,	$\gamma = 1$	1.09
$p = 1$,	$\gamma = 1$	1.33	$p = 1$,	$\gamma = 1$	1.19	$p = 2$,	$\gamma = \infty$	1.07
$p = 5$,	$\gamma = \infty$	1.05	$p = 5$,	$\gamma = \infty$	1.03	$p = 2$,	$\gamma = 1$	1.06
$p = 5$,	$\gamma = 1$	1.09	$p = 5$,	$\gamma = 1$	1.05	$p = 5$,	$\gamma = \infty$	1.03+
						$p = 5$,	$\gamma = 1$	1.03−

Linear Detector

Exponential covariance function			Gaussian covariance function			Linear covariance function		
$p = 0$,	$\gamma = \infty$	2.25	$p = 0$,	$\gamma = \infty$	1.48	$p = 0$,	$\gamma = \infty$	1.59
$p = 0$,	$\gamma =$ 2NG	1.86	$p = 0$,	$\gamma = 2$	1.42	$p = 0$,	$\gamma =$ 1LD	1.28
$p = 0$,	$\gamma =$ LD	1.55	$p = 0$,	$\gamma = 1.5$	1.19	$p = 0$,	$\gamma =$ 1NG	1.18
$p = 0$,	$\gamma = 1$	1.25	$p = 0$,	$\gamma = 1$	1.15	$p = 0$,	$\gamma = .5$	1.08
$p = 0$,	$\gamma = .5$	1.11	$p = 0$,	$\gamma = .5$	1.09	$p = 0$,	$\gamma = .2$	1.05
$p = .5$,	$\gamma = \infty$	1.44	$p = .5$,	$\gamma = \infty$	1.12	$p = 1$,	$\gamma = \infty$	1.13
$p = .5$,	$\gamma = 1$	1.37	$p = .5$,	$\gamma = 1$	1.08	$p = 1$,	$\gamma = 1$	1.14
$p = 2$,	$\gamma = \infty$	1.12	$p = 2$,	$\gamma = \infty$	1.06	$p = 2$,	$\gamma = \infty$	1.05
$p = 2$,	$\gamma = 1$	1.10	$p = 2$,	$\gamma = 1$	1.08	$p = 2$,	$\gamma = 1$	1.08

*The entries in the table are the ratio of the bandwidth at the detector output to the bandwidth at the detector input.
NG means "nearly gaussian"; LD means "low density"

between gaussian and non-gaussian noise. Since the bandwidth is larger as the covariance function is smaller, the results are just opposite to those of the previous section.

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Appendix A

Nearly normal distributions in terms of parametric derivatives. An expansion of a nearly normal distribution in a series of derivatives with respect to the second moments is useful when the noise ensemble has a phase uniformly distributed, independent of the envelope. As has been pointed out in Sec. 2.3, nearly normal narrow-band noise waves have such a structure.

The terms of fixed degree, $M = m + n$, in the moment expansion of the second-order characteristic function (Eq. (3)) are

$$\sum_{\substack{m,n \\ m+n=M}} \frac{\langle R_1^m R_2^n \rangle}{m!\,n!} \langle \cos^m (\omega_0 t_0 - \theta_1) \cos^n (\omega_0[t_0 + t] - \theta_2) \rangle (i\xi_1)^m (i\xi_2)^n$$
$$= \frac{1}{M!} \langle [i\xi_1 R_1 \cos (\omega_0 t_0 - \theta_1) + i\xi_2 R_2 \cos (\omega_0[t_0 + t] - \theta_2)]^M \rangle. \tag{A.1}$$

The average over $\omega_0 t_0 - \theta_1$ can be carried out as a contour integral around the unit circle, after the usual substitution, $z = \exp \{i(\omega_0 t_0 - \theta_1)\}$. There is one singularity inside the contour, a pole of order $M + 1$ at the origin; the integral is zero if M is odd, and, when M is even, gives

$$\frac{\langle \cos^M \theta \rangle}{M!} \langle [(i\xi_1 R_1)^2 + (i\xi_2 R_2^2) + 2(i\xi_1 R_1)(i\xi_2 R_2) \cos (\omega_0 t - \theta)]^{M/2} \rangle. \tag{A.2}$$

For a non-random phase change, to which we here restrict ourselves, the expansion is found compactly through the formal use of tensor notation including the summation convention.

Let the vector $\{X_i\}$, $(i = 1, 2, 3)$ be defined as

$$\{X_i\} = \{R_1^2 , R_1 R_2 , R_2^2\}, \tag{A.3a}$$

and let

$$\{\Xi_i\} = \{\xi_1^2 , 2\xi_1\xi_2 \cos (\omega_0 t - \alpha), \xi_2^2\}, \tag{A.3b}$$

$$\{D_i\} = \left\{\frac{\partial}{\partial\psi_1} , \frac{\partial}{\partial\psi r_0} , \frac{\partial}{\partial\psi_2}\right\}, \tag{A.3c}$$

so that the expression (A.2) becomes, with $M = 2p$,

$$\frac{(-1)^p}{(2p)!} \langle \cos^{2p} \theta \rangle \langle (X_i \Xi_i)^p \rangle = \frac{(-1)^p}{(2_p)!} \langle \cos^{2p} \theta \rangle \langle X_{n_i} \cdots X_{n_p} \rangle \Xi_{n_1} \cdots \Xi_{n_p} . \tag{A.4}$$

The first factors of (A.4) can be recognized as an envelope factor for the moments [from Eq. (12)] so that we can write

$$\frac{(-1)^p}{(2p)!} \mathfrak{M}_{n_1 \cdots n_p} \Xi_{n_1} \cdots \Xi_{n_p} , \tag{A.5}$$

where

$$\mathfrak{M}_{n_1 \cdots n_p} = \langle \cos^{2p} \theta \rangle \langle R_1^{3p - \Sigma n_i} R_2^{\Sigma n_i - p} \rangle = M_{3p - \Sigma n_i , \Sigma n_i - p} .$$

Thus the moment expansion of the characteristic function, which in general is a double series [see Eq. (3)], can be written as a single series when the noise is narrow band,

$$F_2(\xi_1 , \xi_2 ; t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} \mathfrak{M}_{n_1 \cdots n_p} \Xi_{n_1} \cdots \Xi_{n_p} . \tag{A.6}$$

By comparing terms with the semi-invariant expansion of the characteristic function, one finds that, similarly,

$$F_2(\xi_1 , \xi_2 ; t) = \exp\left\{ \sum_{q=1}^{\infty} \frac{(-1)^q}{(2q)!} L_{n_1 \cdots n_q} \Xi_{n_1} \cdots \Xi_{n_q} \right\} . \tag{A.7}$$

where

$$L_{n_1 \cdots n_q} = \Lambda_{3q - \Sigma n_i , \Sigma n_i - q} ,$$

and, in particular, $L_{ij} = \Lambda_{4-i-j, i+j-2}$, $L_{ijk} = \Lambda_{9-i-j-k, i+j+k-3}$, $(i, j, k = 1, 2, 3)$.

If the exponential in Eq. (A.7) is expanded in its power series except for its first term, it is readily seen that each Ξ_i may be replaced by a $2D_i$, giving the desired expansion whose initial terms are

$$F_2(\xi_1, \xi_2; t) = \left\{1 + \frac{2^2}{4!} L_{ij} D_i D_j + \frac{2^3}{6!} L_{ijk} D_i D_j D_k \right.$$
$$\left. + \frac{1}{2} \frac{2^4}{4!^2} L_{ij} L_{kl} D_i D_j D_k D_l + 0(\gamma^{-3}) \right\} \exp \{-\tfrac{1}{2} L_i \Xi_i\} \Bigg|_{\psi_1 = \psi = \psi_2}, \tag{A.8}$$

from which the manner of formation of further terms is evident.