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GEOMETRIC INTERPRETATION FOR THE RECIPROCAL DEFORMATION TENSORS*

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In a finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$, changes of infinitesimal lengths may be measured by the tensor $\mathbf{C}$, where

$$ds^2 = g_{km}\, dx^k\, dx^m = C_{KM}\, dX^K\, dX^M, \qquad C_{KM} = g_{km} x^k_{,K} x^m_{,M}\ , \tag{1}$$

or by the dual tensor $\mathbf{c}$ satisfying formulae that follow by systematic interchange of majuscules and minuscules. Geometric interpretations of $\mathbf{C}$ and $\mathbf{c}$ have been given by Cauchy and others. In 1894 Finger introduced the reciprocal tensors $\mathbf{C}^{-1}$ and $\mathbf{c}^{-1}$, and recent exact work on isotropic elastic bodies employs them often. While formulae such as

$$(C^{-1})^{KM} = g^{km} X^K_{,k} X^M_{,m} \tag{2}$$

for their expression and use are known, geometric interpretation has been lacking.

As is known, the correspondence between elements of area is given by $da^{km} = x^k_{,K}\, x^m_{,M}\, dA^{KM}$, where dA^{KM} is connected with the usual vector element of area dA_K by $dA_K = e_{KMP} dA^{MP}$, $e_{KMP} \equiv (\det G_{QR})^{1/2}\, \epsilon_{KMP}$. Hence[1]

$$\begin{aligned}
(da)^2 &= e^k_{pq} e_{krs} x^p_{,P} x^q_{,Q} x^r_{,R} x^s_{,S}\, dA^{PQ}\, dA^{RS}, \\
&= \frac{\det g_{uv}}{\det G_{UV}}\, g^{km} (\tfrac{1}{2}\epsilon_{krs}\epsilon^{KRS} x^r_{,R} x^s_{,S})(\tfrac{1}{2}\epsilon_{mpq}\epsilon^{MPQ} x^p_{,P} x^q_{,Q})\, dA_K\, dA_M\ , \\
&= \frac{\det g_{uv}}{\det G_{UV}} \left[\frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)}\right]^2 g^{km} X^K_{,k} X^M_{,m}\, dA_K\, dA_M\ , \\
&= [\det (C^{-1})^P_Q]^{-1} (C^{-1})^{KM}\, dA_K\, dA_M\ .
\end{aligned} \tag{3}$$

Comparing this result with (1) shows that the tensor $\mathbf{C}^{-1}/\det \mathbf{C}^{-1}$ measures changes of the magnitudes of infinitesimal areas in precisely the same way as $\mathbf{C}$ measures changes of infinitesimal lengths.

A known principle of duality, which may be called the *first* principle of duality,

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*Received Dec. 4, 1956. This work was done under a National Science Foundation grant to Indiana University.

[1]A formula which is essentially the next to last step in (3) was given by Tonolo, Rend. sem. mat. Padova **14**, 43-117 (1943), Sec. V. 4, but he did not mention any connection with $\mathbf{C}^{-1}$.

enables us to interchange the roles of $\mathbf{X}$ and $\mathbf{x}$ and thus obtain an analogous interpretation for $\mathbf{c}^{-1}/\det \mathbf{c}^{-1}$.

These results are equivalent to a *second principle of duality*: Any proposition on changes of length expressed in terms of $\mathbf{C}$ and $\mathbf{c}$ yields a theorem on changes of area if $\mathbf{C}$, $\mathbf{c}$, and "length" be replaced by $\mathbf{C}^{-1}/\det \mathbf{C}^{-1}$, $\mathbf{c}^{-1}/\det \mathbf{c}^{-1}$, and "area", respectively.

Of the many theorems that may be derived in this way, I record only one: The elements of area suffering extremal changes are normal to the principal directions of strain, and the greatest (least) change of area occurs in the plane normal to the axis of least (greatest) stretch; in fact, if the principal stretches dx/dX satisfy $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$ the corresponding ratios da/dA satisfy $\lambda_2\lambda_3 \leqq \lambda_3\lambda_1 \leqq \lambda_1\lambda_2$. While this theorem is geometrically plausible, the first part does not seem obvious.

BOOK REVIEWS

(*Continued from p. 394*)

The theory of linear antennas. By Ronald W. P. King. Harvard University Press, Cambridge, 1956. xxi + 944 pp. $20.00.

The basic material in this treatise on the linear antenna is founded on a graduate course given by the author, and also includes both the theoretical and experimental data of many other workers in the antenna field. Although much of the information has been published previously in various technical journals, the detailed and extensive compilation and evaluation of so much of this work may be regarded as a worthwhile contribution to the antenna specialist. It is the belief of this reviewer, however, that the general usefulness might have been greatly enhanced if the material had been separated into two or more books.

The stated purpose of this book, which is basically mathematical in its approach, is to provide a bridge from the mathematician to the practical antenna engineer. An introduction summarizing the highlights in the historical development of the linear antenna theory is followed by a short chapter on the essentials of electromagnetic theory.

The mathematical difficulties in treating the behavior of a single linear radiator as end-load for a two-wire line are considered in chapter II. Only antennas having a cross section which is small compared with the wavelength are investigated in this text. Particular attention is paid to the end effect complications and cross-coupling between the line and the antenna. An approximate method of compensating for these effects is developed which employs appropriate lumped reactive elements at the junction between the line and the antenna. The characteristics of the isolated antenna are then studied in detail using several different formulations of the problem. The antenna impedance and admittance calculations are presented in a number of different types of graphical plots as well as in useful tables of numerical values. Unfortunately, in this section of particular interest to the practical engineer, a number of typographical errors occur in identifying the graphs.

Comparisons are made between theoretical calculations and experimental measurements obtained from various sources, and the constructional difficulties involved in precise measuring systems are discussed in considerable detail. In one reference in which this reviewer participated, however, it is noted that the author was incorrect in his statements describing the procedure.

Chapter III is devoted to a general investigation of the mutual coupling between antennas in various geometric configurations. An analysis is made of some of the more common types of antennas such as coplanar arrays, parasitic elements, folded dipoles, V-antennas, asymmetrically driven antennas, etc.

Chapter IV is devoted to the general analysis of the essential properties of receiving and scattering antennas. The freespace patterns and gains of various types of linear radiators are taken up in chapter V. Chapter VI discusses the electromagnetic fields of various configurations of linear radiators in commonly-used arrays.

Chapter VII is devoted to a study of the primary electromagnetic field and radiation characteristics