MULTIPLE FOURIER ANALYSIS IN RECTIFIER PROBLEMS*

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Abstract. The non-linear problem of the multiple Fourier analysis of the output from a cut-off power law rectifier responding to a several-frequency input is reviewed for the one- and two-frequency problems and is briefly investigated for the three-frequency problem. The solutions for the modulation product amplitudes or multiple Fourier coefficients are obtained in exact although transcendental form. An account of the mathematical properties of these multiple Fourier coefficients or Bennett functions, including hypergeometric representations and power series expansions for them as well as recurrence relations satisfied by them, is given in the paper together with line graphs of the first ten basic functions for the one-frequency problem and of the first fifteen basic functions for the two-frequency problem. Further applications of the theory are also given to the computation of average output power with the aid of the multiple Fourier coefficients or Bennett functions studied in the paper, and the work is concluded with some brief remarks concerning the interpretation of the results in terms of the theory of almost periodic functions and the generalized Fourier series of Bohr under appropriate conditions. Numerical tables of the functions graphed have been prepared and are available separately in the United States and Great Britain for applications requiring great accuracy. Finally, the entire theory is based on the original method of the expansion of the rectifier output in multiple Fourier series introduced by Bennett in 1933 and 1947.

1. Introduction. A non-linear problem of continuing interest in theoretical electronics, and one much studied in the one- and two-frequency instances, is the multiple Fourier analysis of the output from a cut-off power law rectifier responding to a several-frequency input under various assumptions about the cut-off bias, about the exponent of the power law, and about the input amplitude ratios. Among others Bennett [1, 2], Kaufman [3, 4], Lampard [5], Salzberg [6], and Sternberg, Kaufman, Shipman, and Thurston [7, 8, 9, 10, 11] have obtained exact although transcendental solutions for the modulation product amplitudes or multiple Fourier coefficients in the first two cases mentioned and have tabulated some of these quantities under the name of Bennett functions, while Feuerstein [12], following a method of Bennett [1] and Rice [13], has recently given a quite extensive treatment of the n-frequency problem in terms of generalized Weber-Schafheitlin integrals [14] suitable for numerical evaluation.

In this paper, after reviewing some of the more interesting results noted above for the one- and two-frequency problems associated with a cut-off power law rectifier, with one or two new results added, we briefly investigate the three-frequency problem by a transcendental method of extension of previous results using, particularly, certain integral recurrence relations which hold between the multiple Fourier coefficients or Bennett functions of different multiplicities. Since the frequencies and phase angles are readily found, the real problem with which we are concerned in each instance consists of evaluating the single, double, or triple integrals defining the Fourier coefficients in the multiple Fourier series expansion of the output from the rectifier. The mathematical properties of these multiple Fourier coefficients or Bennett functions are investigated

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systematically, and hypergeometric representations and power series expansions for them and recurrence relations satisfied by them are given in the paper, together with line graphs of the first ten basic functions for the one-frequency problem and of the first fifteen basic functions for the two-frequency problem. In addition to providing for the multiple Fourier series expansion of the output from the rectifier, it is shown that the average output power may also be computed in terms of the same multiple Fourier coefficients or Bennett functions as those studied, provided, in the two- and three-frequency cases, that the input is non-periodic. No attempt is made to discuss modulation product problems more general than those associated with rectifiers, the reader in such instances being referred to Sternberg and Kaufman [7, 8, 9] for a general approximation process applicable to any continuous modulator, the solution again being formulated in terms of Bennett functions as here. The paper is concluded with some brief remarks concerning the interpretation of the results in terms of the theory of almost periodic functions and the generalized Fourier series of Bohr, again under the proviso, in the two- and three-frequency cases, that the input be non-periodic. Numerical tables giving values of the functions graphed good to 1×10^{-6} units at values of the arguments spaced one-tenth of a unit apart have been prepared and are available separately from the sources noted below in the United States and Great Britain for applications requiring great accuracy. Finally, we may note in passing that the entire theory is based on the original method of the expansion of the rectifier output in multiple Fourier series introduced by Bennett [1, 2] in 1933 and 1947.

The general rectifier problem is formulated precisely and the solution in simple or multiple Fourier series is outlined in Sec. 2. The review of results for the one- and two-frequency problems is presented in Secs. 3 and 4 while the three-frequency problem is treated in Sec. 5. The results of the theory which apply to the computation of average output power for a single frequency periodic input or a multiple frequency non-periodic input are given in Sec. 6, and the concluding remarks bearing on the connections with the theory of almost periodic functions and the generalized Fourier series of Bohr are given in Sec. 7. Copies of the numerical tables of the Bennett functions graphed in the paper and available to the public have been deposited with Mr. D. H. Lehmer of the Unpublished Mathematical Tables File in the United States and with Mr. Andrew Young of the Grace Library of the University of Liverpool in Great Britain and with the authors.

2. Formulation of the problem. Consider a cut-off power law rectifier having an output versus input characteristic $Y = Y^\nu(X; X_0)$ of the form

$$Y^\nu(X; X_0) = \begin{cases} (X - X_0)^\nu, & X > X_0, \\ 0, & X \leq X_0, \end{cases} \qquad \nu \geq 0. \tag{2.1}$$

Let the input to the rectifier be a one-, two-, or three-frequency function of time, $x(t)$, of one of the forms

$$x(t) = P \cos (pt + \theta_p), \qquad P > 0, \tag{2.2}$$

$$x(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q), \qquad P \geq Q > 0, \tag{2.3}$$

or

$$x(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q) + R \cos (rt + \theta_r), \qquad P \geq Q \geq R > 0. \tag{2.4}$$

The output from the rectifier, $y(t) \equiv Y^{\nu}[x(t); X_0]$ has then a single, double, or triple Fourier series expansion of the corresponding form

$$y(t) = \tfrac{1}{2}P^{\nu}A_0^{(\nu)}(h) + P^{\nu}\sum_{m=1}^{\infty} A_m^{(\nu)}(h)\cos(\omega_m t + \phi_m), \tag{2.5}$$

$$y(t) = \tfrac{1}{2}P^{\nu}A_{00}^{(\nu)}(h, k) + P^{\nu}\sum_{m,n=0}^{\infty}{}^{*} A_{\pm mn}^{(\nu)}(h, k)\cos(\omega_{\pm mn}t + \phi_{\pm mn}), \tag{2.6}$$

or

$$y(t) = \tfrac{1}{2}P^{\nu}A_{000}^{(\nu)}(h, k_1, k_2) + P^{\nu}\sum_{m,n,l=0}^{\infty}{}^{*} A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2)\cos(\omega_{\pm\pm mnl}t + \phi_{\pm\pm mnl}), \tag{2.7}$$

where $h = X_0/P$, $k = Q/P > 0$, and $k_1 = Q/P \geq k_2 = R/P > 0$, and where the modulation product angular frequencies, phase angles, and amplitudes are given by the formulas

$$\omega_m = mp, \qquad \phi_m = m\theta_p, \tag{2.8}$$

$$\omega_{\pm mn} = mp \pm nq, \qquad \phi_{\pm mn} = m\theta_p \pm n\theta_q, \tag{2.9}$$

$$\omega_{\pm\pm mnl} = mp \pm nq \pm lr, \qquad \phi_{\pm\pm mnl} = m\theta_p \pm n\theta_q \pm l\theta_r, \tag{2.10}$$

and the formulas

$$A_m^{(\nu)}(h) = \frac{2}{\pi}\int_{\mathcal{R}_1} (\cos u - h)^{\nu}\cos mu\, du, \qquad \mathcal{R}_1 : \cos u \geq h, \quad 0 \leq u \leq \pi, \tag{2.11}$$

$$A_{\pm mn}^{(\nu)}(h, k) = \frac{2}{\pi^2}\iint_{\mathcal{R}_2} (\cos u + k\cos v - h)^{\nu}\cos mu\, du\cos nv\, dv, \qquad \mathcal{R}_2 : \cos u + k\cos v \geq h, \quad 0 \leq u, v \leq \pi, \tag{2.12}$$

$$A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2) = \frac{2}{\pi^3}\iiint_{\mathcal{R}_3} (\cos u + k_1\cos v + k_2\cos w - h)^{\nu} \cdot \cos mu\, du\cos nv\, dv\cos lw\, dw, \qquad \mathcal{R}_3 : \cos u + k_1\cos v + k_2\cos w \geq h, \quad 0 \leq u, v, w \leq \pi, \tag{2.13}$$

where in all of these formulas the indices m, n, l take all integral values $m, n, l \geq 0$ and, finally, where the asterisks on the summation signs in (2.6) and (2.7) indicate that in these multiple Fourier series we sum only on all distinct arrangements of plus and minus signs, equivalent arrangements being taken only with the plus signs and with the zero order terms, particularly, having been removed from the sums.

We will assume the genesis of these Fourier expansions to be well-known and will refer the reader to Bennett [1] or Sternberg and Kaufman [7] for additional details of such matters. It may be worth noting, however, that the existence of these Fourier series follows at once from the continuity of the kernel functions in the integrands of the quantities $A_m^{(\nu)}(h)$, $A_{\pm mn}^{(\nu)}(h, k)$, and $A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2)$ within the corresponding regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ while the convergence of these Fourier series to the functions $y(t)$ follows

at least at all points of continuity of the $y(t)$, in the case of (2.5) from the elementary theory, in case of (2.6) from the theory of the double Fourier series given by Hobson [15] or Tonelli [16] and, finally, in the case of (2.7) may presumably be surmised from Hobson's remarks concerning comparative properties of the double and multiple Fourier series in general.

As noted in the introduction the whole problem of multiple harmonic analysis with which we are concerned for rectifiers consists of the evaluation of the multiple Fourier coefficients or the integrals $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ in (2.11), (2.12), and (2.13) as functions of their several parameters. Aside from certain relatively simple closed form solutions obtainable in the one-frequency problem none of these integrals can be evaluated in closed form in terms of elementary functions, except for specialized values of the parameters h, k, k_1, and k_2, but rather in general the best that can be obtained are solutions in a transcendental form of one type or another with the final computations to be done by numerical methods. Although one might wish for more simple results in the two- and three-frequency problems, the power of the methods to be presented should not be underestimated for either theoretical or practical purposes. Finally, in recognition of the original work of Bennett [1, 2] and following previous usage, we term the functions $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ Bennett functions of the νth kind and of multiplicities one, two, and three respectively.

3. Review of results for the one-frequency problem. In the one-frequency problem for the rectifier (2.1) we have for the input the signal (2.2) and for the output the Fourier series (2.5) and have to determine particularly the Fourier coefficients

$$A_m^{(\nu)}(h) = \frac{2}{\pi}\int_{\mathcal{R}_1} (\cos u - h)^\nu \cos mu\, du, \qquad \mathcal{R}_1 : \cos u \geq h, \quad 0 \leq u \leq \pi, \tag{3.1}$$

as functions of $h = X_0/P$ for fixed $\nu \geq 0$. The variable h takes all real values and, following Kaufman [3], we note three cases of the functions $A_m^{(\nu)}(h)$ according to the scheme: (0) $h \geq 1$, (a) $|h| \leq 1$, and (∞) $h \leq -1$. Clearly case (0) may be disposed of at once for in this case the rectifier is biased so strongly that we have $A_m^{(\nu)}(h) = 0$ for all m.

Consider now cases (a) and (∞) for all $\nu \geq 0$. To begin with, the functions $A_m^{(\nu)}(h)$ satisfy a number of recurrence relations. Thus, for determining the $(\nu + 1)$th kind functions in terms of those of the νth kind we have the recurrence formulas

$$\begin{aligned} \tfrac{1}{2}A_0^{(\nu+1)} &\equiv \tfrac{1}{2}A_1^{(\nu)} - \tfrac{1}{2}hA_0^{(\nu)}, \\ 2mA_m^{(\nu+1)} &\equiv (\nu + 1)A_{m-1}^{(\nu)} - (\nu + 1)A_{m+1}^{(\nu)}, \end{aligned} \tag{3.2}$$

while for purposes of evaluating the $(m + 1)$th order functions in terms of those of the mth and lower orders we have the recurrence formula

$$(m + \nu + 1)A_{m+1}^{(\nu)} \equiv 2mhA_m^{(\nu)} - (m - \nu - 1)A_{m-1}^{(\nu)}, \tag{3.3}$$

where in each of these relations $m \geq 1$. For proofs we refer to Kaufman [3]. For actual evaluation of the functions $A_m^{(\nu)}(h)$ two general solutions are available. In particular, in case (a) Bennett [2] has given the formula

$$A_m^{(\nu)}(h) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 3/2)} (1 - h)^{\nu+1/2} F[\tfrac{1}{2} + m, \tfrac{1}{2} - m; \nu + \tfrac{3}{2}; \tfrac{1}{2}(1 - h)], \tag{3.4}$$

while in case (∞) Lampard [5] has given the result

$$A_m^{(\nu)}(h) = \frac{2^{1-m}}{m!} \frac{\Gamma(\nu + 1)}{\Gamma(\nu - m + 1)} \mid h \mid^{\nu - m} F[\tfrac{1}{2}(m - \nu), \tfrac{1}{2}(m - \nu + 1); m + 1; h^{-2}], \tag{3.5}$$

where in each of these formulas $F(a, b; c; x)$ denotes the Gaussian hypergeometric function and where $m \geq 0$. Various alternative forms of these solutions have also been noted by the same authors, Bennett, in particular, indicating certain special results for integral and half integral values of ν; see also Salzberg [6].

Consider next cases (a) and (∞) for integral values of ν. In view of the recurrence relations (3.2) for determining the functions $A_m^{(\nu+1)}(h)$ in terms of the functions $A_m^{(\nu)}(h)$, it is clear that for such values of ν the solutions may be completely expressed in terms of those for a basic set such as $A_m^{(0)}(h)$ or $A_m(h) = A_m^{(1)}(h)$. Actually we choose the latter as being more important in their own right. In case (a) following Bennett [2] we may simplify (3.4) or integrate directly with Salzberg [6] to obtain the special results

$$\begin{aligned} \tfrac{1}{2}A_0(h) &= \frac{1}{\pi}[(1 - h^2)^{1/2} - h \cos^{-1} h], \\ A_1(h) &= \frac{1}{\pi}[\cos^{-1} h - h(1 - h^2)^{1/2}], \\ A_m(h) &= \frac{1}{m\pi}\left\{\frac{\sin[(m - 1)\cos^{-1} h]}{m - 1} - \frac{\sin[(m + 1)\cos^{-1} h]}{m + 1}\right\}, \end{aligned} \tag{3.6}$$

where $m \geq 2$, while in case (∞) we have

$$\tfrac{1}{2}A_0(h) = \mid h \mid, \qquad A_1(h) = 1, \qquad A_m(h) = 0, \tag{3.7}$$

where again $m \geq 2$. These formulas represent an essentially complete solution for the functions $A_m(h) = A_m^{(1)}(h)$. For many purposes, however, the corresponding power series expansions are more illuminating. Expanding the elementary functions in (3.6) and combining terms we have, for example, in case (a) for the first few functions $A_m(h) = A_m^{(1)}(h)$ the series

$$\tfrac{1}{2}A_0(h) = \frac{1}{\pi}\left[1 - \frac{\pi}{2}h + \frac{1}{2}h^2 + \frac{1}{24}h^4 + \frac{1}{80}h^6 + \cdots\right], \tag{3.8}$$

$$A_1(h) = \frac{1}{\pi}\left[\frac{\pi}{2} - 2h + \frac{1}{3}h^3 + \frac{1}{20}h^5 + \frac{1}{56}h^7 + \cdots\right], \tag{3.9}$$

$$A_2(h) = \frac{1}{\pi}\left[\frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots\right], \tag{3.10}$$

$$A_3(h) = \frac{h}{\pi}\left[\frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots\right], \tag{3.11}$$

$$A_4(h) = \frac{1}{\pi}\left[-\frac{2}{15} + h^2 - \frac{5}{4}h^4 + \frac{7}{24}h^6 + \cdots\right], \tag{3.12}$$

$$A_5(h) = \frac{h}{\pi}\left[-\frac{2}{5} + \frac{5}{3}h^2 - \frac{7}{4}h^4 + \frac{3}{8}h^6 + \cdots\right], \tag{3.13}$$

each of which converges uniformly for $\mid h \mid \leq 1$ and, indeed quite rapidly, the terms given above, for example, being sufficient to evaluate the functions $\tfrac{1}{2}A_0(h)$ to $A_5(h)$ to

better than 1×10^{-3} units for $|h| \leq \frac{1}{2}$. Finally, in addition to the above formulas and series for the functions $A_m(h) = A_m^{(1)}(h)$ in cases (a) and (∞) Kaufman [3] has given the reflection relations

$$\tfrac{1}{2}A_0(-h) \equiv \tfrac{1}{2}A_0(h) + h, \quad A_1(-h) \equiv 1 - A_1(h), \quad A_m(-h) \equiv (-1)^m A_m(h), \tag{3.14}$$

for determining the functions of negative bias in terms of those of positive bias where again $m \geq 2$.

For elementary applications and general interest line graphs of the first ten functions $\frac{1}{2}A_0(h) = \frac{1}{2}A_0^{(1)}(h)$ and $A_m(h) = A_m^{(1)}(h)$ are presented in Fig. 1 while for applications of greater accuracy the reader may refer to the tables of these functions mentioned in the introduction.

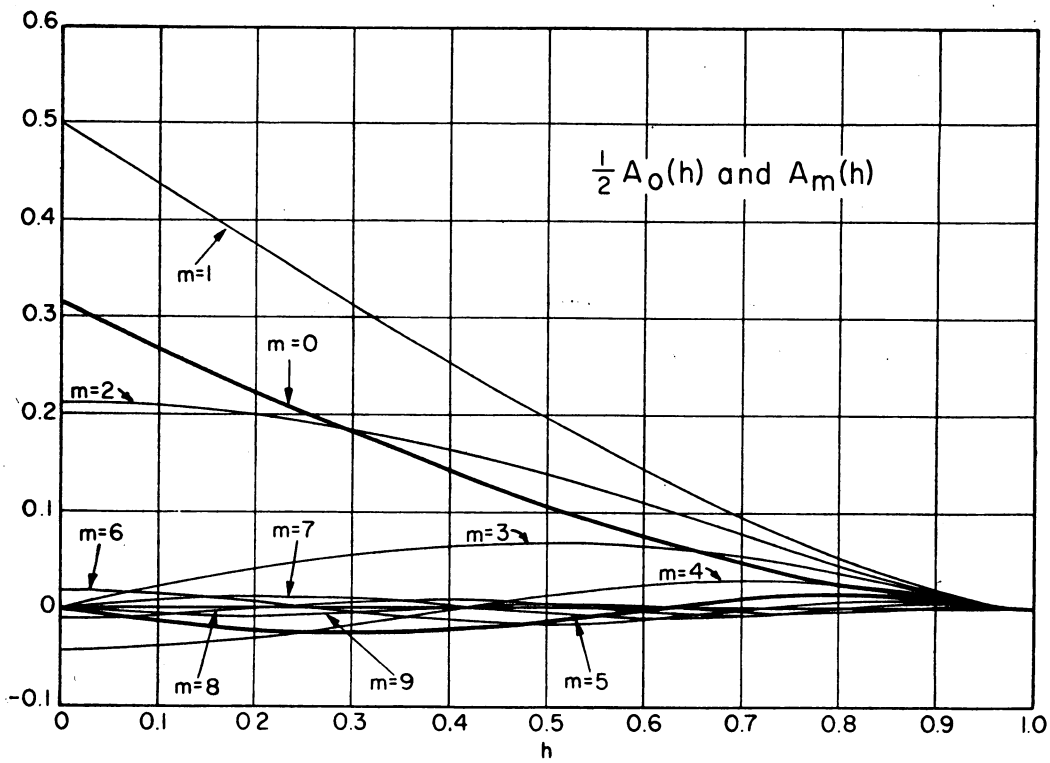

FIG. 1. The functions $\frac{1}{2}A_0(h)$ and $A_m(h, k)$

4. Review of results for the two-frequency problem. In the two-frequency problem for the rectifier (2.1) we have for the input the signal (2.3) and for the output the double Fourier series (2.6) and have to determine particularly the Fourier coefficients

$$A_{mn}^{(\nu)}(h, k) = \frac{2}{\pi^2} \iint_{\mathcal{R}_2} (\cos u + k \cos v - h)^\nu \cos mu \, du \cos nv \, dv,$$
$$\mathcal{R}_2 : \cos u + k \cos v \geq h, \qquad 0 \leq u, v \leq \pi, \tag{4.1}$$

as functions of $h = X_0/P$ and $k = Q/P$ for fixed $\nu \geq 0$. The variable h takes all real values while the variable k ranges over the interval $0 < k \leq 1$ and, as before, we note three cases of the functions $A_{mn}^{(\nu)}(h, k)$ according as: (0) $h \geq 1 + k$, (a) $|h| \leq 1 + k$,

or (∞) $h \leq -1 - k$. Clearly again case (0) may be disposed of at once, the rectifier being biased so strongly that we have $A^{(\nu)}_{\pm mn}(h, k) = 0$ for all m, n while in all cases the functions $A^{(\nu)}_{\pm mn}(h, k)$ are independent of the double sign so that we may drop it hereafter and write simply $A^{(\nu)}_{mn}(h, k)$.

Consider now cases (a) and (∞) for all $\nu \geq 0$. As before the functions $A^{(\nu)}_{mn}(h, k)$ satisfy a number of recurrence relations. Thus, first of all comparing the definitions (3.1) and (4.1) of the functions $A^{(\nu)}_{m}(h)$ and $A^{(\nu)}_{mn}(h, k)$ one readily observes the fundamental formulas

$$\begin{aligned} A^{(\nu)}_{mn}(h, k) &\equiv \frac{1}{\pi}\int_0^\pi A^{(\nu)}_m(h - k\cos v)\cos nv\,dv, \\ A^{(\nu)}_{mn}(h, k) &\equiv \frac{k^\nu}{\pi}\int_0^\pi A^{(\nu)}_n(h' - k'\cos u)\cos mu\,du, \end{aligned} \tag{4.2}$$

where $h' = h/k$ and $k' = 1/k$, which express the $A^{(\nu)}_{mn}(h, k)$ in terms of the $A^{(\nu)}_m(h)$ and, hence, in a sense constitute a complete solution of the two-frequency problem in terms of the solutions of the one-frequency problem. Next, for determining the $(\nu + 1)$th kind of functions in terms of those of the νth kind we have much as before the recurrence formulas

$$\begin{aligned} &\text{(a)} \quad \tfrac{1}{2}A^{(\nu+1)}_{00} \equiv \tfrac{1}{2}A^{(\nu)}_{10} + \tfrac{1}{2}kA^{(\nu)}_{01} - \tfrac{1}{2}hA^{(\nu)}_{00}, \\ &\text{(b)} \quad 2mA^{(\nu+1)}_{mn} \equiv (\nu + 1)A^{(\nu)}_{m-1,n} - (\nu + 1)A^{(\nu)}_{m+1,n}, \\ &\text{(c)} \quad 2nA^{(\nu+1)}_{mn} \equiv (\nu + 1)kA^{(\nu)}_{m,n-1} - (\nu + 1)kA^{(\nu)}_{m,n+1}, \end{aligned} \tag{4.3}$$

where in (b) $m \geq 1$, $n \geq 0$, and in (c) $m \geq 0$, $n \geq 1$, while for purposes of evaluating the $(m + n + 1)$th order functions in terms of those of the $(m + n)$th and lower orders we have the recurrence formulas

$$\begin{aligned} &\text{(a)} \quad (m + n + \nu + 1)A^{(\nu)}_{m+1,n} \equiv 2mhA^{(\nu)}_{mn} \\ &\qquad\qquad - 2mkA^{(\nu)}_{m,n-1} - (m - n - \nu - 1)A^{(\nu)}_{m-1,n}, \\ &\text{(b)} \quad (m - n + \nu + 3)A^{(\nu)}_{m+2,n-1} \equiv 2(m + 1)hA^{(\nu)}_{m+1,n-1} \\ &\qquad\qquad - 2(m + 1)kA^{(\nu)}_{m+1,n} - (m + n - \nu - 1)A^{(\nu)}_{m,n-1}, \\ &\text{(c)} \quad (n + m + \nu + 1)A^{(\nu)}_{m,n+1} \equiv 2nh'A^{(\nu)}_{mn} \\ &\qquad\qquad - 2nk'A^{(\nu)}_{m-1,n} - (n - m - \nu - 1)A^{(\nu)}_{m,n-1}, \\ &\text{(d)} \quad (n - m + \nu + 3)A^{(\nu)}_{m-1,n+2} \equiv 2(n + 1)h'A^{(\nu)}_{m-1,n+1} \\ &\qquad\qquad - 2(n + 1)k'A^{(\nu)}_{m,n+1} - (n + m - \nu - 1)A^{(\nu)}_{m-1,n}, \end{aligned} \tag{4.4}$$

where $h' = h/k$ and $k' = 1/k$ and where in (a) and (c) $m \geq 1$, $n \geq 1$, in (b) $m \geq 0$, $n \geq 1$, and in (d) $m \geq 1$, $n \geq 0$. The proofs of these relations, though computationally lengthy, are entirely elementary and may be effected by combined use of the recurrence relations (3.2) and (3.3) for the functions $A^{(\nu)}_m(h)$ and the formulas (4.2) for expressing the functions $A^{(\nu)}_{mn}(h, k)$ in terms of the functions $A^{(\nu)}_m(h)$; see also Kaufman [4]. For actual evaluation of the functions $A^{(\nu)}_{mn}(h, k)$ only one general solution seems to be available. Thus, in case (∞) Lampard [5] has given the formula

$$\begin{aligned} A^{(\nu)}_{mn}(h, k) = \frac{2^{1-m-n}}{m!n!}\frac{\Gamma(\nu + 1)}{\Gamma(\nu - m - n + 1)}\,|h|^{\nu-m-n}k^n \\ \cdot F_4[\tfrac{1}{2}(m + n - \nu), \tfrac{1}{2}(m + n - \nu + 1); m + 1, n + 1; h^{-2}, k^2h^{-2}], \end{aligned} \tag{4.5}$$

where $F_4(a, b; c, d; x, y)$ denotes Appell's fourth type of hypergeometric function of two variables [17] and where $m \geq 0$, $n \geq 0$. Other forms of this solution have also been noted by the same writer.

Consider next cases (a) and (∞) for integral values of ν. Similarly as before the recurrence relations (4.3) for determining the functions $A_{mn}^{(\nu+1)}(h, k)$ in terms of the functions $A_{mn}^{(\nu)}(h, k)$ make it clear that for such values of ν the solutions may be completely expressed in terms of those for a basic set such as $A_{mn}^{(0)}(h, k)$ or $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ and again we make the latter choice. Even for these functions it is not possible to give a complete solution free from integral signs or numerical procedures in case (a). However, in the subcase of case (a) in which $|h| + k \leq 1$, using a method amounting to substitution of the series (3.8) to (3.11) and so on for the functions $A_m(h) = A_m^{(1)}(h)$ into the formulas (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$, Sternberg and Kaufman [7] have derived double power series expansions of the forms

$$\frac{1}{2} A_{00}(h, k) = -\frac{1}{2} h + \frac{1}{\pi} \left\{ \begin{array}{l} 1 + \frac{1}{2} h^2 + \frac{1}{24} h^4 + \frac{1}{80} h^6 + \cdots \\ + \frac{1}{4} k^2 + \frac{1}{8} h^2 k^2 + \frac{3}{32} h^4 k^2 + \cdots \\ + \frac{1}{64} k^4 + \frac{9}{128} h^2 k^4 + \cdots \\ + \frac{1}{256} k^6 + \cdots \\ \cdots \end{array} \right\}, \tag{4.6}$$

$$A_{10}(h, k) = \frac{1}{2} + \frac{h}{\pi} \left\{ \begin{array}{l} -2 + \frac{1}{3} h^2 + \frac{1}{20} h^4 + \frac{1}{56} h^6 + \cdots \\ + \frac{1}{2} k^2 + \frac{1}{4} h^2 k^2 + \frac{3}{16} h^4 k^2 + \cdots \\ + \frac{3}{32} k^4 + \frac{15}{64} h^2 k^4 + \cdots \\ + \frac{5}{128} k^6 + \cdots \\ \cdots \end{array} \right\}, \tag{4.7}$$

$$A_{01}(h, k) = \frac{k}{2} + \frac{hk}{\pi} \left\{ \begin{array}{l} -1 - \frac{1}{6} h^2 - \frac{3}{40} h^4 - \frac{5}{112} h^6 - \cdots \\ - \frac{1}{8} k^2 - \frac{3}{16} h^2 k^2 - \frac{15}{64} h^4 k^2 - \cdots \\ - \frac{3}{64} k^4 - \frac{25}{128} h^2 k^4 - \cdots \\ - \frac{25}{1024} k^6 - \cdots \\ \cdots \end{array} \right\}, \tag{4.8}$$

$$A_{20}(h, k) = \frac{1}{\pi}\left\{\begin{array}{l} \frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots \\ -\frac{1}{2}k^2 + \frac{3}{4}h^2k^2 + \frac{5}{16}h^4k^2 + \cdots \\ +\frac{3}{32}k^4 + \frac{15}{64}h^2k^4 + \cdots \\ +\frac{5}{384}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.9}$$

$$A_{11}(h, k) = \frac{k}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{2}h^2 - \frac{1}{8}h^4 - \frac{1}{16}h^6 - \cdots \\ -\frac{1}{8}k^2 - \frac{3}{16}h^2k^2 - \frac{15}{16}h^4k^2 - \cdots \\ -\frac{1}{64}k^4 - \frac{15}{128}h^2k^4 - \cdots \\ -\frac{5}{1024}k^6 - \cdots \\ \cdots \end{array}\right\}, \tag{4.10}$$

$$A_{02}(h, k) = \frac{k^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{8}h^2 + \frac{3}{32}h^4 + \frac{5}{64}h^6 + \cdots \\ +\frac{1}{48}k^2 + \frac{3}{32}h^2k^2 + \frac{25}{128}h^4k^2 + \cdots \\ +\frac{3}{512}k^4 + \frac{75}{1024}h^2k^4 + \cdots \\ +\frac{5}{2048}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.11}$$

$$A_{30}(h, k) = \frac{h}{\pi}\left\{\begin{array}{l} \frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots \\ -\frac{3}{2}k^2 + \frac{5}{4}h^2k^2 + \frac{7}{16}h^4k^2 + \cdots \\ +\frac{15}{32}k^4 + \frac{35}{64}h^2k^4 + \cdots \\ +\frac{35}{384}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.12}$$

$$A_{21}(h, k) = \frac{hk}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{2}h^2 - \frac{1}{8}h^4 - \frac{1}{16}h^6 - \cdots \\ -\frac{3}{8}k^2 - \frac{5}{16}h^2k^2 - \frac{21}{64}h^4k^2 - \cdots \\ -\frac{5}{64}k^4 - \frac{35}{128}h^2k^4 - \cdots \\ -\frac{35}{1024}k^6 - \cdots \\ \cdots \end{array}\right\}, \tag{4.13}$$

$$A_{12}(h, k) = \frac{hk^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{8}h^2 + \frac{3}{32}h^4 + \frac{5}{64}h^6 + \cdots \\ +\frac{1}{16}k^2 + \frac{5}{32}h^2k^2 + \frac{35}{128}h^4k^2 + \cdots \\ +\frac{15}{512}k^4 + \frac{175}{1024}h^2k^4 + \cdots \\ +\frac{35}{2048}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.14}$$

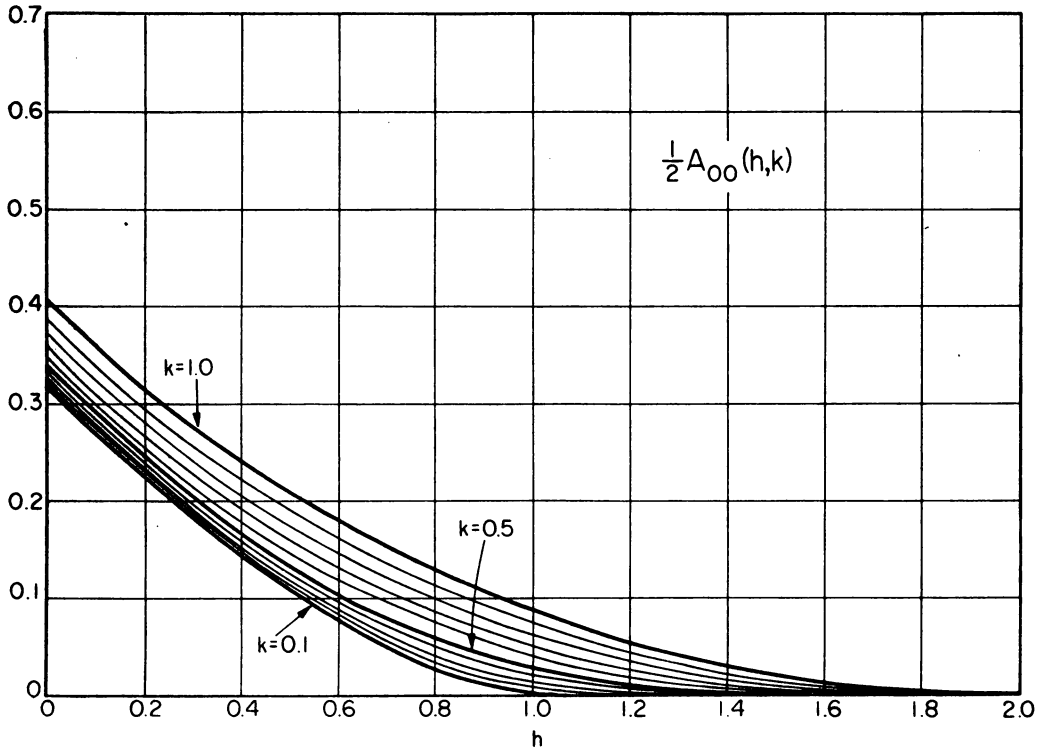

FIG. 2. The function $\frac{1}{2}A_{00}(h, k)$

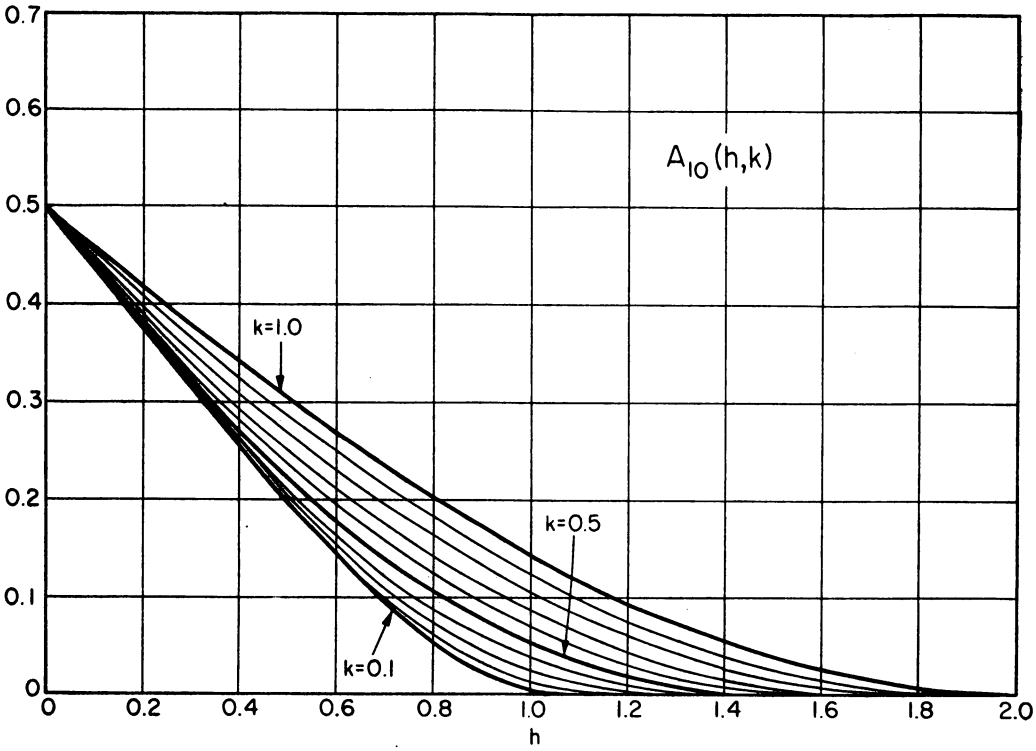

FIG. 3. The function $A_{10}(h, k)$

$$A_{03}(h, k) = \frac{hk^3}{\pi}\left\{\begin{array}{l} -\frac{1}{24} - \frac{1}{16}h^2 - \frac{5}{64}h^4 - \frac{35}{384}h^6 - \cdots \\ -\frac{3}{128}k^2 - \frac{25}{256}h^2k^2 - \frac{245}{1024}h^4k^2 - \cdots \\ -\frac{15}{1024}k^4 - \frac{1445}{12288}h^2k^4 - \cdots \\ -\frac{245}{24576}k^6 - \cdots \\ \cdots \end{array}\right\}, \quad (4.15)$$

and so forth, each of which converges uniformly for $|h| + k \leq 1$ and, in fact, quite rapidly, the terms given above, for example, being sufficient to evaluate the functions $\frac{1}{2}A_{00}(h, k)$ to $A_{03}(h, k)$ to better than 1×10^{-3} units for $|h| + k \leq \frac{1}{2}$. In case (∞) as before we have

$$\begin{aligned} \tfrac{1}{2}A_{00}(h, k) &= |h|, \qquad & A_{10}(h, k) &= 1, \\ A_{01}(h, k) &= k, \qquad & A_{mn}(h, k) &= 0, \end{aligned} \quad (4.16)$$

where $m + n \geq 2$. For half-integral values of ν in the sub-case of case (a) in which $|h| + k \leq 1$ similar double power series expansions for the functions $A_{mn}^{(\nu)}(h, k)$ also may be obtained by combined application of the formulas (3.4) for the functions $A_m^{(\nu)}(h)$

and (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$, the half-integral property of ν making possible the explicit computation of the coefficients in the series, while in case (∞) the formula (4.5) for the functions $A_{mn}^{(\nu)}(h, k)$ also applies with some slight simplifications of the gamma functions. Finally, in addition to the above formulas and series for the functions $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ in cases (a) and (∞) Bennett [2] and Sternberg and Kaufman [7] have given the reflection relations

$$\begin{aligned} \tfrac{1}{2}A_{00}(-h, k) &\equiv \tfrac{1}{2}A_{00}(h, k) + h, & A_{10}(-h, k) &\equiv 1 - A_{10}(h, k), \\ A_{01}(-h, k) &\equiv k - A_{01}(h, k), & A_{mn}(-h, k) &\equiv (-1)^{m+n}A_{mn}(h, k), \end{aligned} \tag{4.17}$$

for determining the functions of negative bias in terms of those of positive bias where again $m + n \geq 2$. These formulas also may be derived in an elementary manner with the aid of previous results for the functions $A_m(h) = A_m^{(1)}(h)$ and the formulas (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$.

For elementary applications and general usefulness line graphs of the first fifteen functions $\frac{1}{2}A_{00}(h, k) = \frac{1}{2}A_{00}^{(1)}(h, k)$ and $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ are presented in Figs. 2 to 16 while for applications requiring greater accuracy the reader as before may refer to the tables of these functions described in the introduction.

5. Some new results for the three-frequency problem. In the three-frequency problem for the rectifier (2.1) we have for the input the signal (2.4) and for the output the triple Fourier series (2.7) and have now to determine the Fourier coefficients

$$\begin{aligned} A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2) = \frac{2}{\pi^3} \iiint_{\mathfrak{R}_3} (\cos u + k_1 \cos v + k_2 \cos w - h)^\nu \\ \cdot \cos mu\, du \cos nv\, dv \cos lw\, dw, \end{aligned} \tag{5.1}$$

$$\mathfrak{R}_3 : \cos u + k_1 \cos v + k_2 \cos w \geq h, \qquad 0 \leq u, v, w \leq \pi,$$

as functions of $h = X_0/P$, $k_1 = Q/P$, and $k_2 = R/P$ for fixed $\nu \geq 0$. The variable h again takes all real values while the variables k_1 and k_2 range over the intervals $0 < k_2 \leq k_1 \leq 1$ and, as before, there are three cases of the functions $A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2)$ according as: (0) $h \geq 1 + k_1 + k_2$, (a) $|h| \leq 1 + k_1 + k_2$, or (∞) $h \leq -1 - k_1 - k_2$. Similarly as before in case (0) we have $A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2) = 0$ for all m, n, l while in all cases the functions $A_{\pm\pm mnl}^{(\nu)}(h, k_1, k_2)$ are independent of the double signs so that we may write simply $A_{mnl}^{(\nu)}(h, k_1, k_2)$.

Consider now cases (a) and (∞) for all $\nu \geq 0$. As before the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$ satisfy a number of recurrence relations. Thus, to begin with comparing the definitions (4.1) and (5.1) of the functions $A_{mn}^{(\nu)}(h, k)$ and $A_{mnl}^{(\nu)}(h, k_1, k_2)$, one readily observes in a manner similar to that previously used the fundamental formulas

$$\begin{aligned} A_{mnl}^{(\nu)}(h, k_1, k_2) &\equiv \frac{1}{\pi}\int_0^\pi A_{mn}^{(\nu)}(h - k_2 \cos w, k_1) \cos lw\, dw, \\ A_{mnl}^{(\nu)}(h, k_1, k_2) &\equiv \frac{1}{\pi}\int_0^\pi A_{ml}^{(\nu)}(h - k_1 \cos v, k_2) \cos nv\, dv, \\ A_{mnl}^{(\nu)}(h, k_1, k_2) &\equiv \frac{k_1^\nu}{\pi}\int_0^\pi A_{nl}^{(\nu)}(h_1' - k_1' \cos u, k_1'k_2) \cos mu\, du, \end{aligned} \tag{5.2}$$

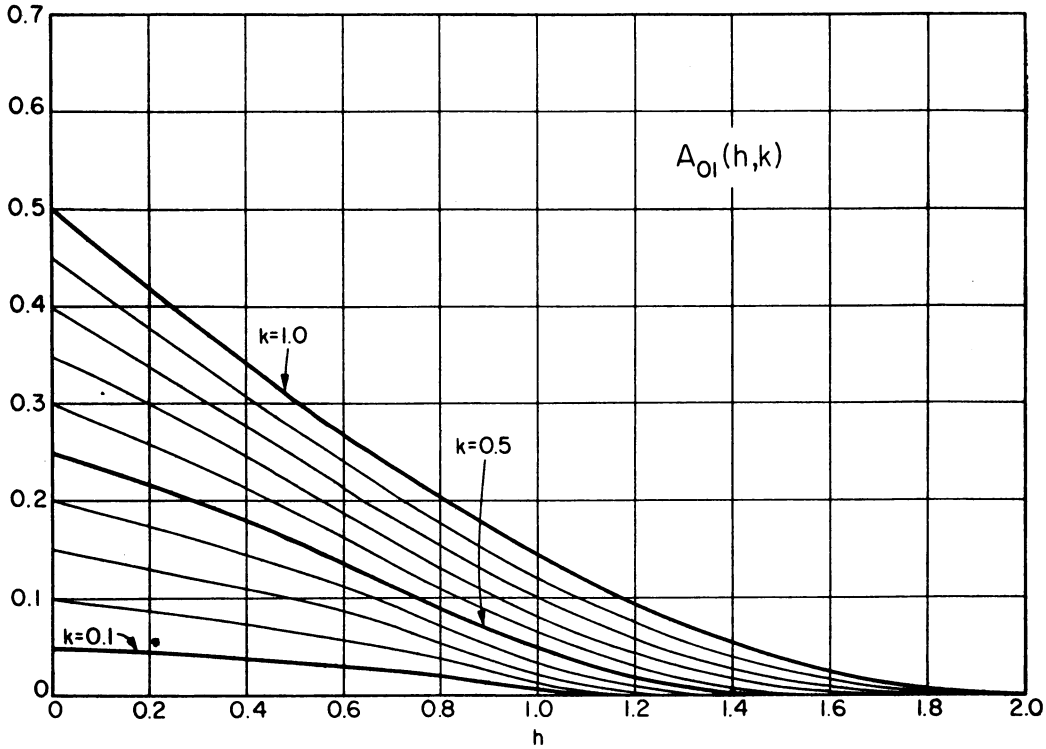

FIG. 4. The function $A_{01}(h, k)$

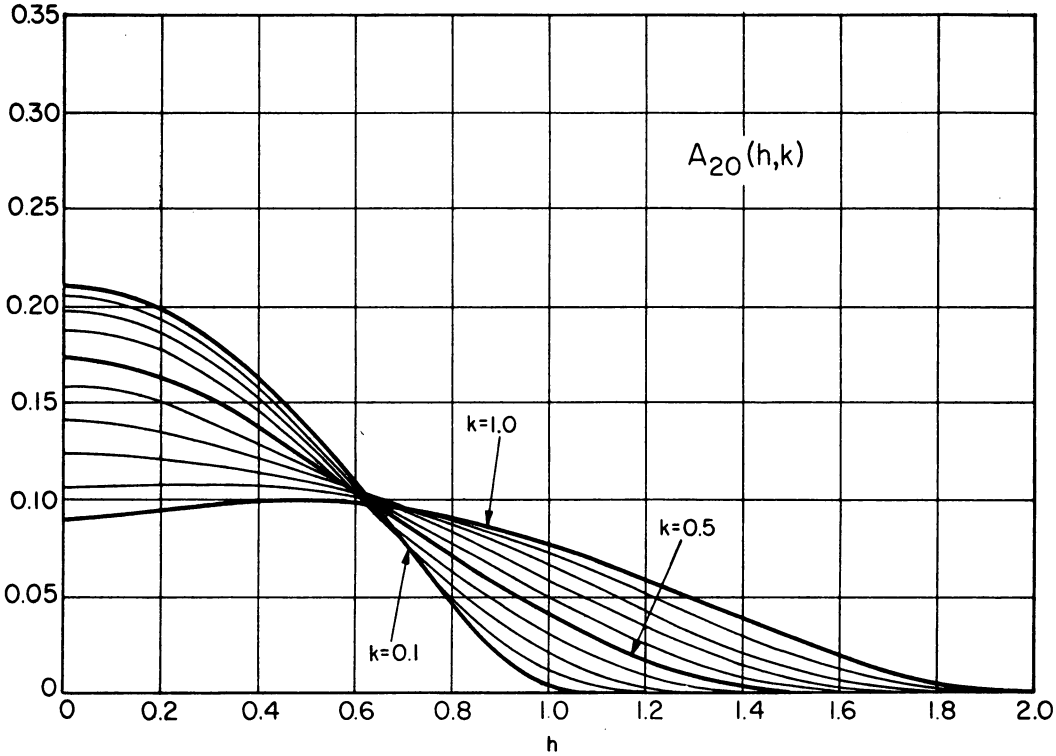

Fig. 5. The function $A_{20}(h, k)$

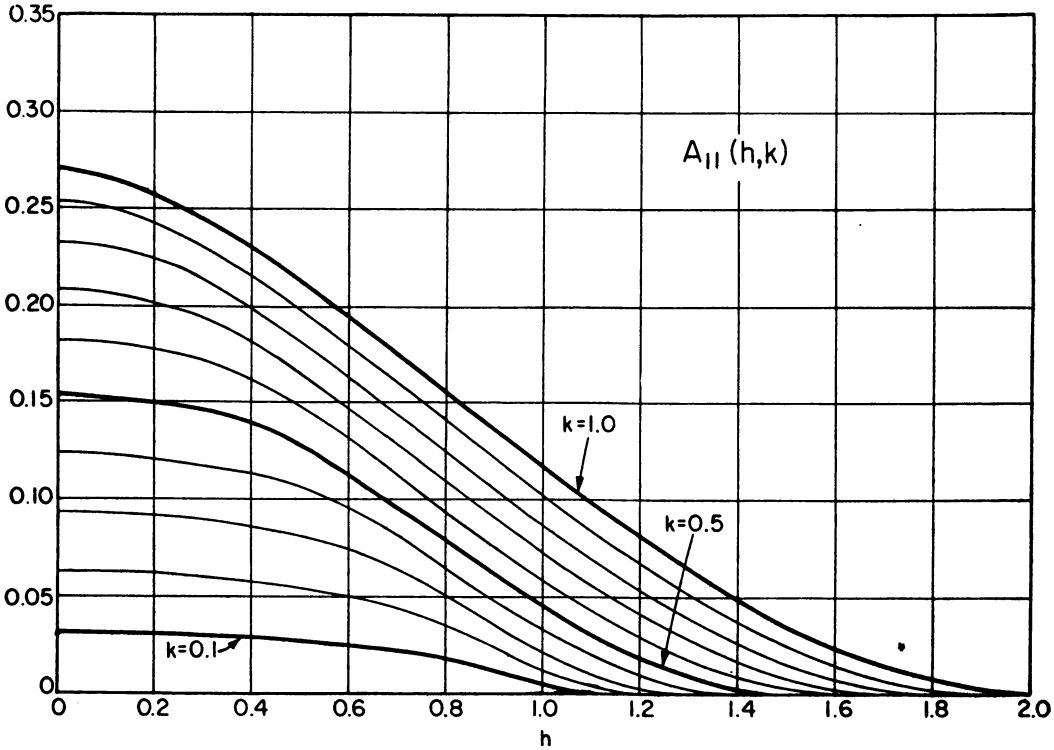

Fig. 6. The function $A_{11}(h, k)$

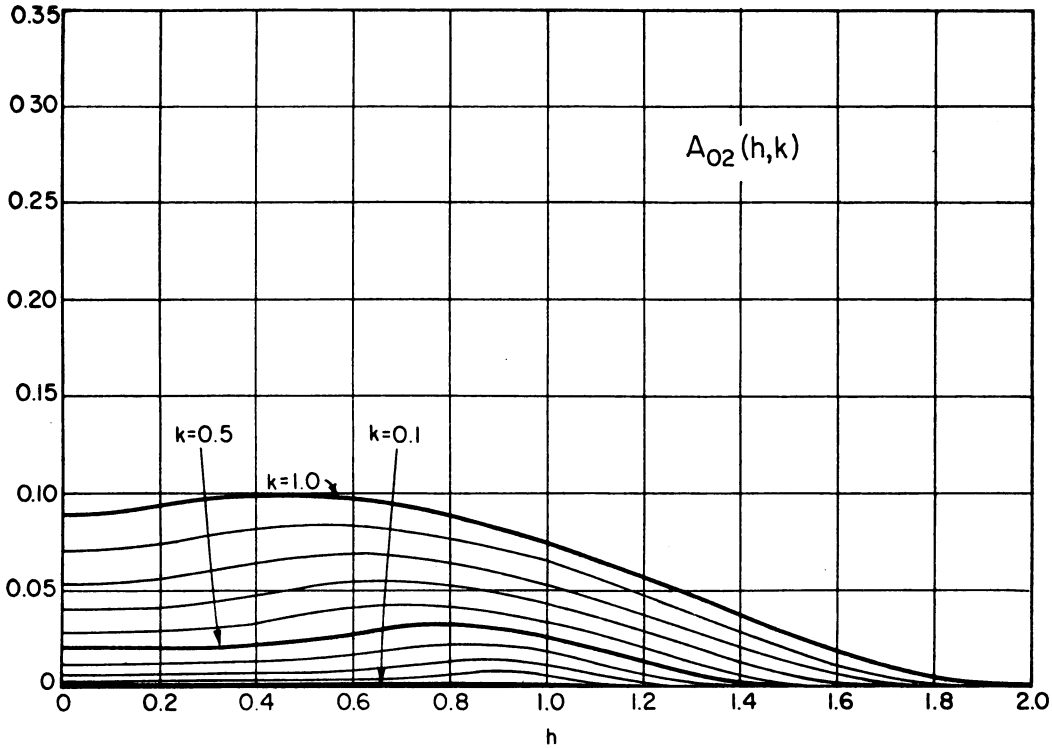

FIG. 7. The function $A_{02}(h, k)$

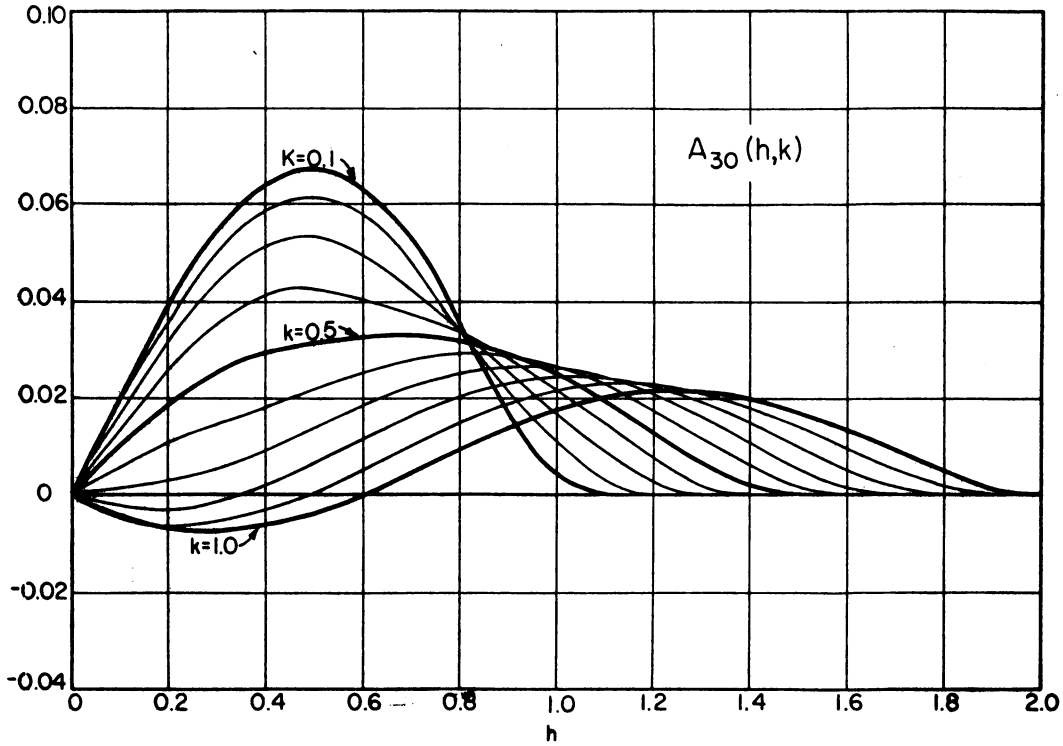

FIG. 8. The function $A_{30}(h, k)$

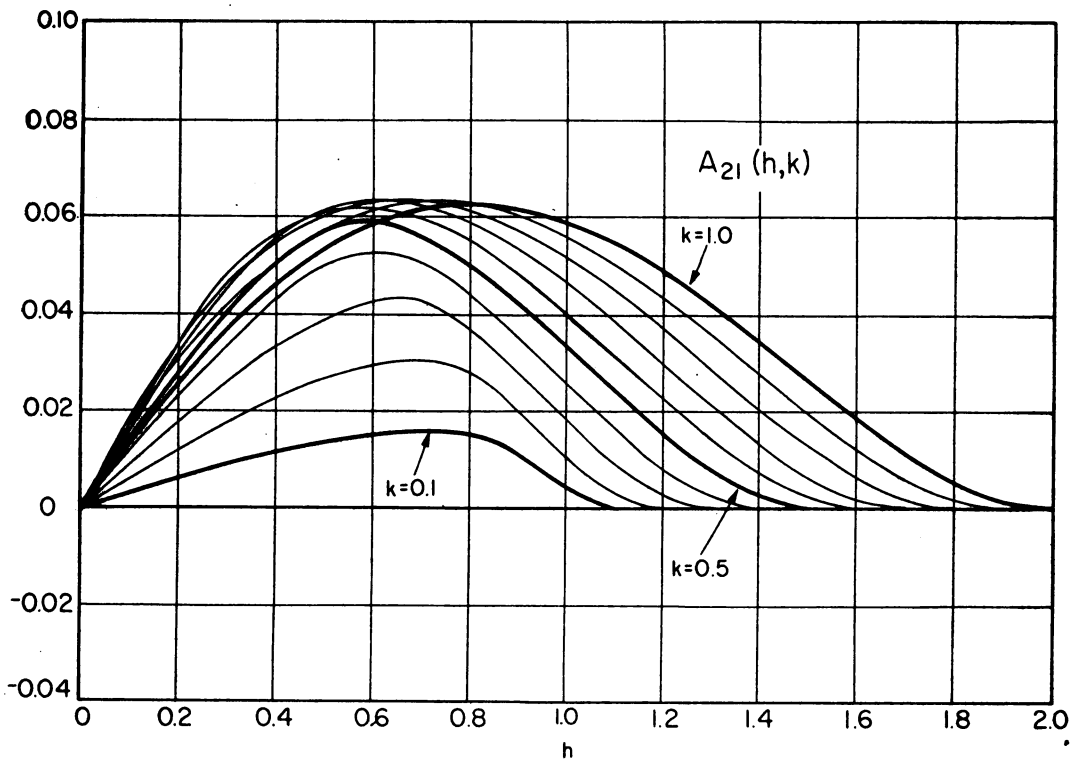

FIG. 9. The function $A_{21}(h, k)$

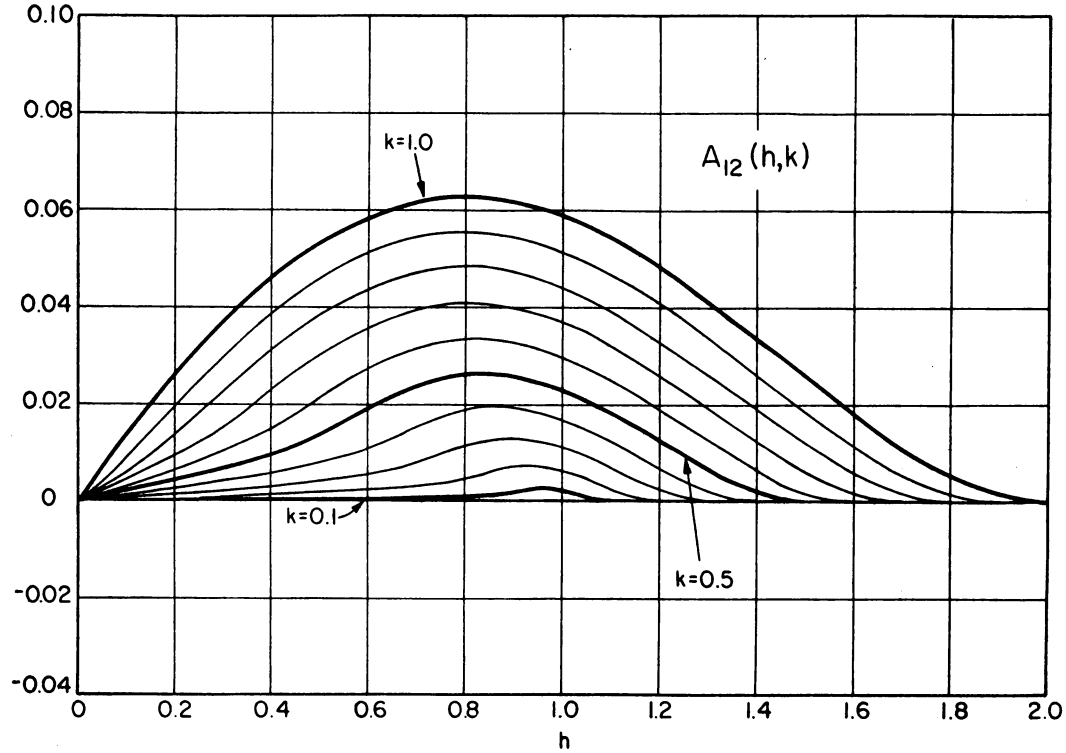

FIG. 10. The function $A_{12}(h, k)$

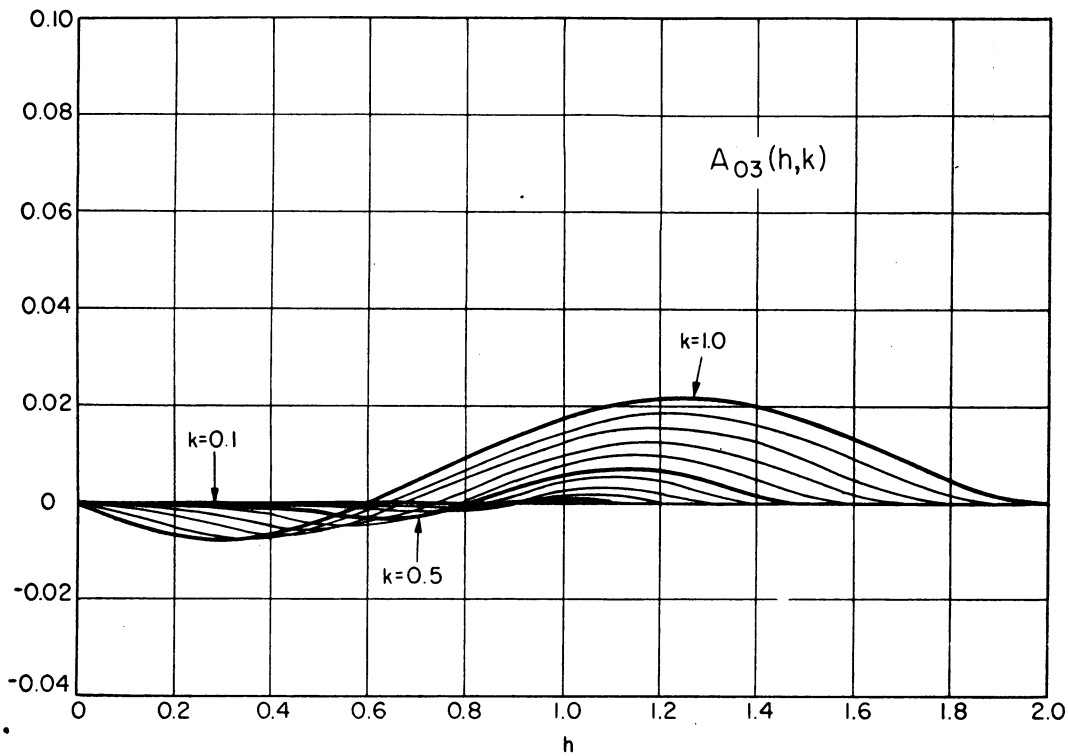

FIG. 11. The function $A_{03}(h, k)$

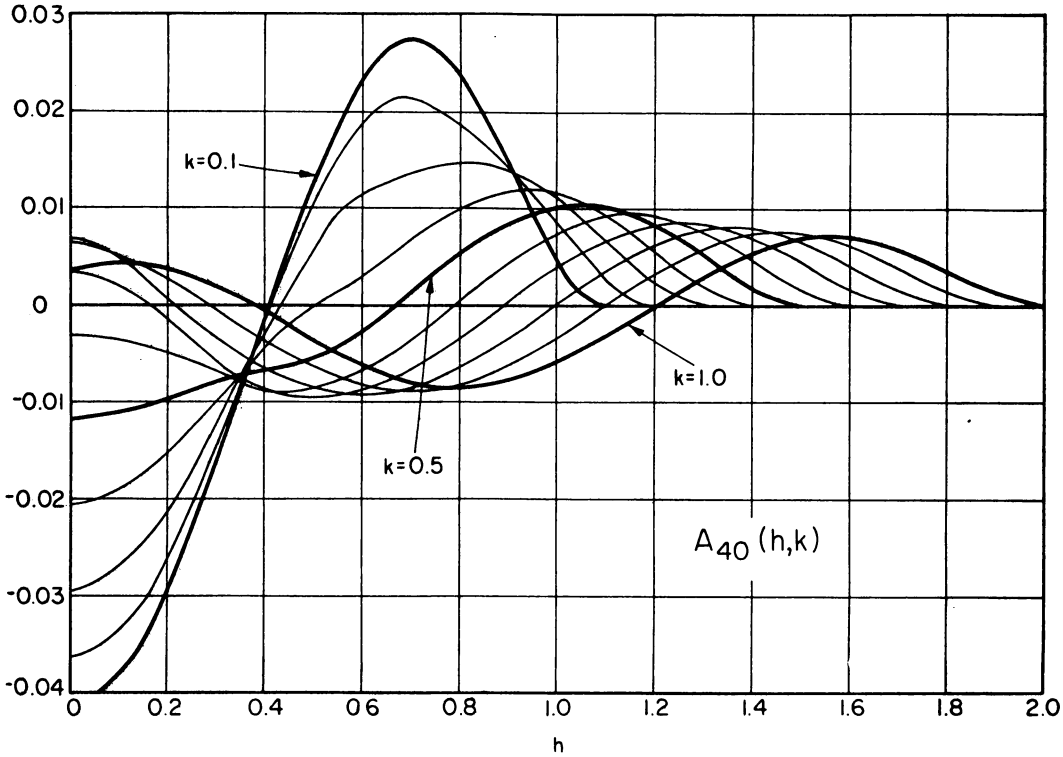

FIG. 12. The function $A_{40}(h, k)$

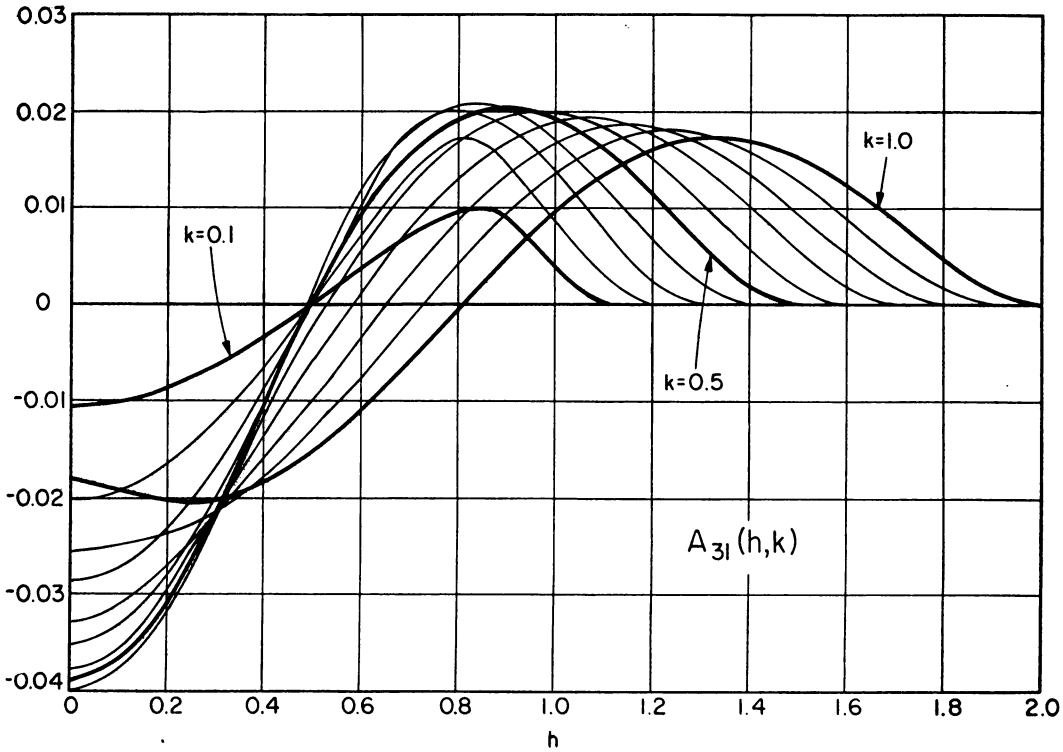

FIG. 13. The function $A_{31}(h, k)$

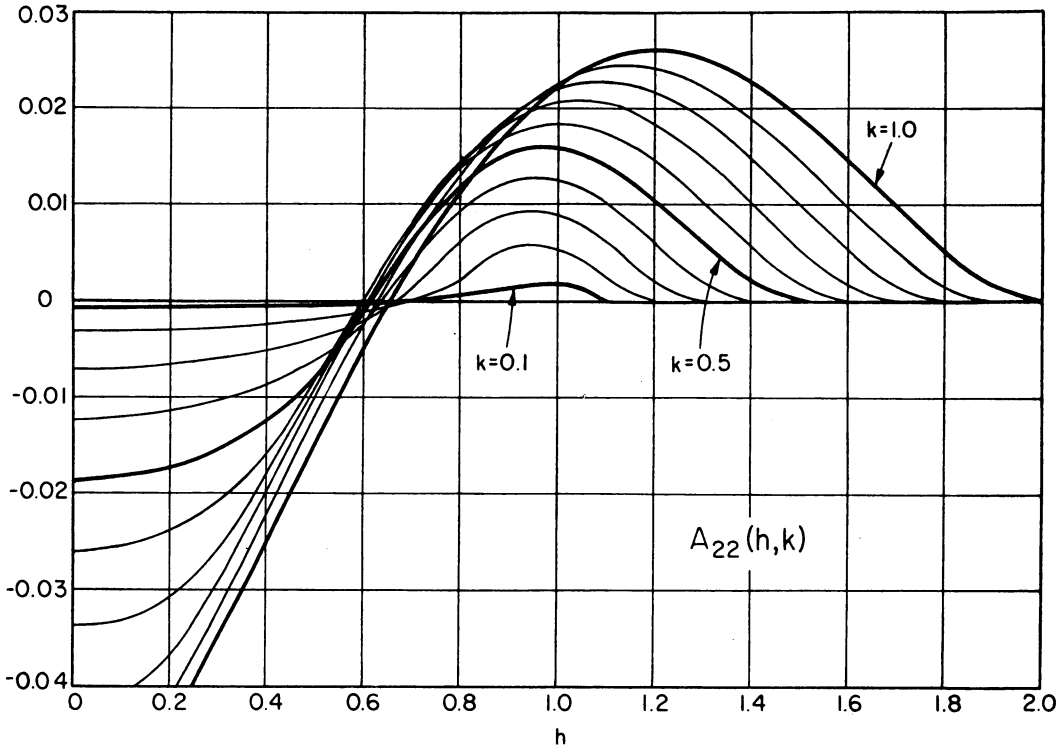

FIG. 14. The function $A_{22}(h, k)$

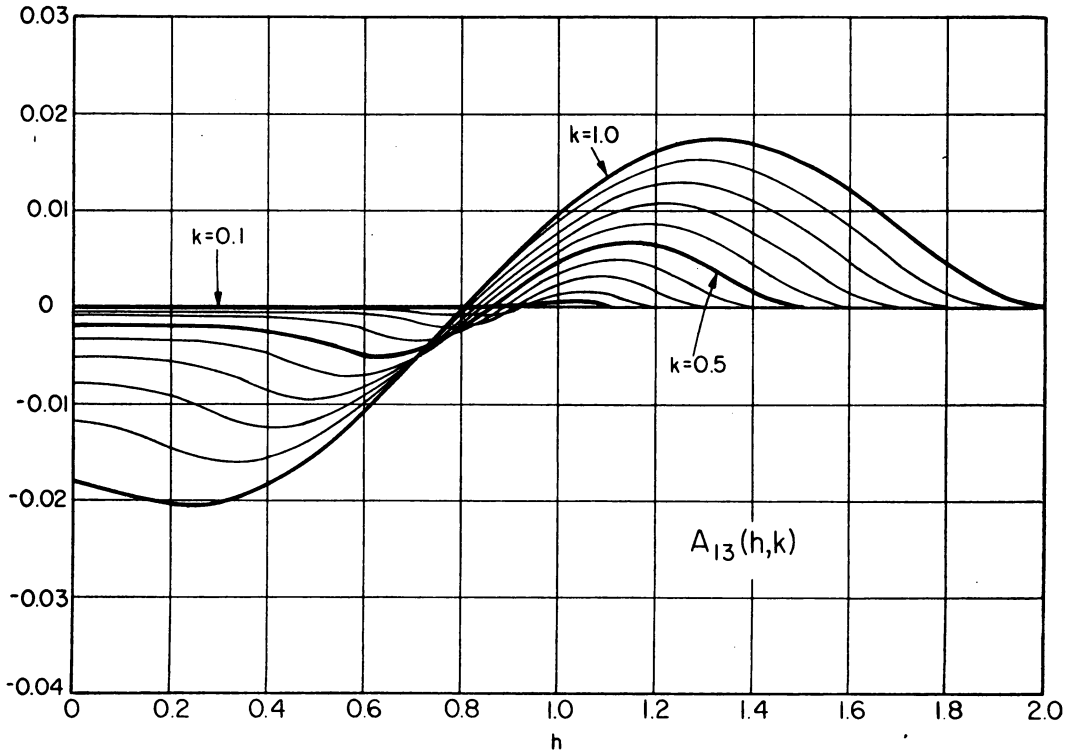

Fig. 15. The function $A_{13}(h, k)$

where $h_1' = h/k_1$ and $k_1' = 1/k_1$, which express the $A_{mnl}^{(\nu)}(h, k_1, k_2)$ in terms of the $A_{mn}^{(\nu)}(h, k)$ and, hence, in a sense constitute a complete solution of the three-frequency problem in terms of the solutions of the two-frequency problem. Next, for determining the higher kind and higher order functions in terms of those of the νth kind and $(m + n + l)$th order we have respectively the recurrence formulas

$$\begin{aligned}
&\text{(a)} && \tfrac{1}{2}A_{000}^{(\nu+1)} \equiv \tfrac{1}{2}A_{100}^{(\nu)} + \tfrac{1}{2}k_1A_{010}^{(\nu)} + \tfrac{1}{2}k_2A_{001}^{(\nu)} - \tfrac{1}{2}hA_{000}^{(\nu)}, \\
&\text{(b)} && 2mA_{mnl}^{(\nu+1)} \equiv (\nu + 1)A_{m-1,n,l}^{(\nu)} - (\nu + 1)A_{m+1,n,l}^{(\nu)}, \\
&\text{(c)} && 2nA_{mnl}^{(\nu+1)} \equiv (\nu + 1)k_1A_{m,n-1,l}^{(\nu)} - (\nu + 1)k_1A_{m,n+1,l}^{(\nu)}, \\
&\text{(d)} && 2lA_{mnl}^{(\nu+1)} \equiv (\nu + 1)k_2A_{m,n,l-1}^{(\nu)} - (\nu + 1)k_2A_{m,n,l+1}^{(\nu)},
\end{aligned} \tag{5.3}$$

where in (b) $m \geq 1, n \geq 0, l \geq 0$, in (c) $m \geq 0, n \geq 1, l \geq 0$, and in (d) $m \geq 0, n \geq 0, l \geq 1$, and the recurrence formulas

$$\begin{aligned}
&\text{(a)} && (m + n + l + \nu + 1)A_{m+1,n,l}^{(\nu)} \equiv 2mhA_{mnl}^{(\nu)} \\
& && \quad - 2mk_1A_{m,n-1,l}^{(\nu)} - 2mk_2A_{m,n,l-1}^{(\nu)} \\
& && \quad - (m - n - l - \nu - 1)A_{m-1,n,l}^{(\nu)}, \\
&\text{(b)} && (m - n + l + \nu + 3)A_{m+2,n-1,l}^{(\nu)} \equiv 2(m + 1)hA_{m+1,n-1,l}^{(\nu)} \\
& && \quad - 2(m + 1)k_1A_{m+1,n,l}^{(\nu)} - 2(m + 1)k_2A_{m+1,n-1,l-1}^{(\nu)} \\
& && \quad - (m + n - l - \nu - 1)A_{m,n-1,l}^{(\nu)}, \\
&\text{(c)} && (m + n - l + \nu + 3)A_{m+2,n,l-1}^{(\nu)} \equiv 2(m + 1)hA_{m+1,n,l-1}^{(\nu)} \\
& && \quad - 2(m + 1)k_2A_{m+1,n,l}^{(\nu)} - 2(m + 1)k_1A_{m+1,n-1,l-1}^{(\nu)} \\
& && \quad - (m - n + l - \nu - 1)A_{m,n,l-1}^{(\nu)}, \\
&\text{(d)} && (m - n - l + \nu + 5)A_{m+3,n-1,l-1}^{(\nu)} \equiv 2(m + 2)hA_{m+2,n-1,l-1}^{(\nu)} \\
& && \quad - 2(m + 2)k_2A_{m+2,n-1,l}^{(\nu)} - 2(m + 2)k_1A_{m+2,n,l-1}^{(\nu)} \\
& && \quad - (m + n + l - \nu - 1)A_{m+1,n-1,l-1}^{(\nu)},
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
&\text{(a)} && (n + l + m + \nu + 1)A_{m,n+1,l}^{(\nu)} \equiv 2nh_1'A_{mnl}^{(\nu)} \\
& && \quad - 2nk_1'k_2A_{m,n,l-1}^{(\nu)} - 2nk_1'A_{m-1,n,l}^{(\nu)} \\
& && \quad - (n - l - m - \nu - 1)A_{m,n-1,l}^{(\nu)}, \\
&\text{(b)} && (n - l + m + \nu + 3)A_{m,n+2,l-1}^{(\nu)} \equiv 2(n + 1)h_1'A_{m,n+1,l-1}^{(\nu)} \\
& && \quad - 2(n + 1)k_1'k_2A_{m,n+1,l}^{(\nu)} - 2(n + 1)k_1'A_{m-1,n+1,l-1}^{(\nu)} \\
& && \quad - (n + l - m - \nu - 1)A_{m,n,l-1}^{(\nu)}, \\
&\text{(c)} && (n + l - m + \nu + 3)A_{m-1,n+2,l}^{(\nu)} \equiv 2(n + 1)h_1'A_{m-1,n+1,l}^{(\nu)} \\
& && \quad - 2(n + 1)k_1'A_{m,n+1,l}^{(\nu)} - 2(n + 1)k_1'k_2A_{m-1,n+1,l-1}^{(\nu)} \\
& && \quad - (n - l + m - \nu - 1)A_{m-1,n,l}^{(\nu)}, \\
&\text{(d)} && (n - l - m + \nu + 5)A_{m-1,n+3,l-1}^{(\nu)} \equiv 2(n + 2)h_1'A_{m-1,n+2,l-1}^{(\nu)} \\
& && \quad - 2(n + 2)k_1'A_{m,n+2,l-1}^{(\nu)} - 2(n + 2)k_1'k_2A_{m-1,n+2,l}^{(\nu)} \\
& && \quad - (n + l + m - \nu - 1)A_{m-1,n+1,l-1}^{(\nu)},
\end{aligned} \tag{5.5}$$

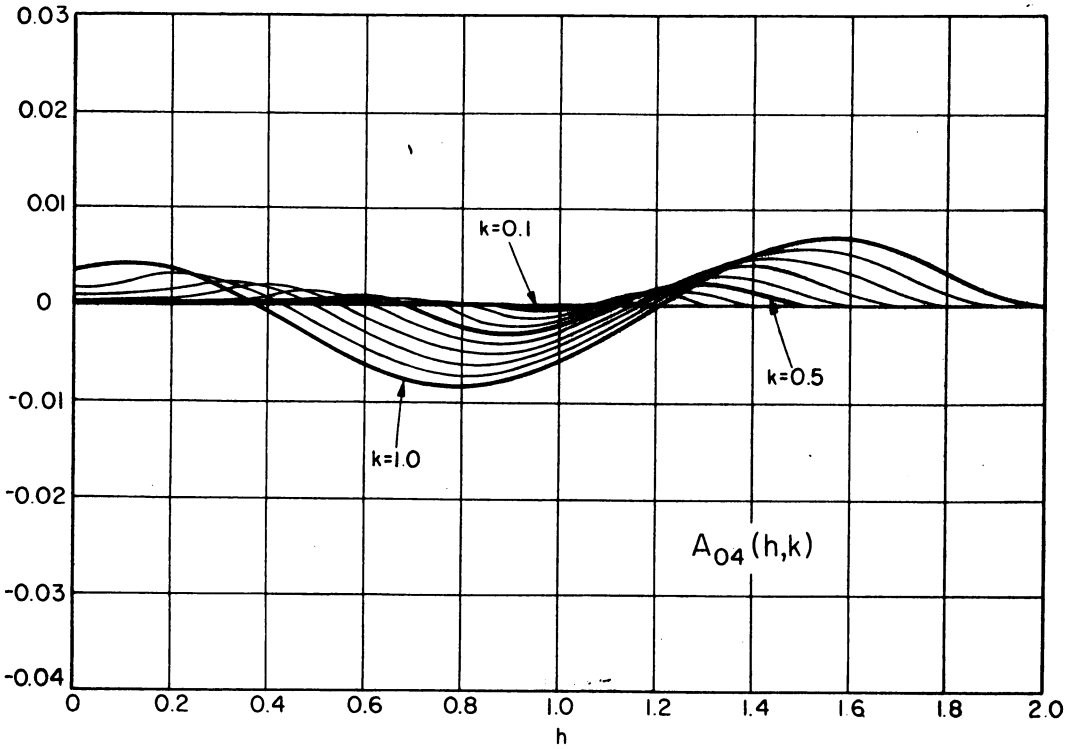

FIG. 16. The function $A_{04}(h, k)$

and

$$
\begin{aligned}
&\text{(a)} \quad (l+m+n+\nu+1)A^{(\nu)}_{m,n,l+1} \equiv 2lh_2'A^{(\nu)}_{m,n,l} \\
&\qquad\qquad - 2lk_2'A^{(\nu)}_{m-1,n,l} - 2lk_1k_2'A^{(\nu)}_{m,n-1,l} \\
&\qquad\qquad - (l-m-n-\nu-1)A^{(\nu)}_{m,n,l-1}\,, \\
&\text{(b)} \quad (l-m+n+\nu+3)A^{(\nu)}_{m-1,n,l+2} \equiv 2(l+1)h_2'A^{(\nu)}_{m-1,n,l+1} \\
&\qquad\qquad - 2(l+1)k_2'A^{(\nu)}_{m,n,l+1} - 2(l+1)k_1k_2'A^{(\nu)}_{m-1,n-1,l+1} \\
&\qquad\qquad - (l+m-n-\nu-1)A^{(\nu)}_{m-1,n,l}\,, \\
&\text{(c)} \quad (l+m-n+\nu+3)A^{(\nu)}_{m,n-1,l+2} \equiv 2(l+1)h_2'A^{(\nu)}_{m,n-1,l+1} \\
&\qquad\qquad - 2(l+1)k_1k_2'A^{(\nu)}_{m,n,l+1} - 2(l+1)k_2'A^{(\nu)}_{m-1,n-1,l+1} \\
&\qquad\qquad - (l-m+n-\nu-1)A^{(\nu)}_{m,n-1,l}\,, \\
&\text{(d)} \quad (l-m-n+\nu+5)A^{(\nu)}_{m-1,n-1,l+3} \equiv 2(l+2)h_2'A^{(\nu)}_{m-1,n-1,l+2} \\
&\qquad\qquad - 2(l+2)k_1k_2'A^{(\nu)}_{m-1,n,l+2} - 2(l+2)k_2'A^{(\nu)}_{m,n-1,l+2} \\
&\qquad\qquad - (l+m+n-\nu-1)A^{(\nu)}_{m-1,n-1,l+1}\,,
\end{aligned}
\tag{5.6}
$$

where $h_1' = h/k_1$, $h_2' = h/k_2$, $k_1' = 1/k_1$, and $k_2' = 1/k_2$ and where in formulas (5.4)-(a), (5.5)-(a), and (5.6)-(a) $m \geq 1$, $n \geq 1$, $l \geq 1$; in formulas (5.4)-(b) and (c) $m \geq 0$,

$n \geq 1$, $l \geq 1$; in formulas (5.5)–(b) and (c) $m \geq 1$, $n \geq 0$, $l \geq 1$; in formulas (5.6)–(b) and (c) $m \geq 1$, $n \geq 1$, $l \geq 0$ while in formulas (5.4)–(d), (5.5)–(d), and (5.6)–(d) we have respectively $m \geq -1$, $n \geq 1$, $l \geq 1$; $m \geq 1$, $n \geq -1$, $l \geq 1$; and $m \geq 1$, $n \geq 1$, $l \geq -1$. As before the proof of these relations, though computationally lengthy, is entirely elementary and may be carried out by combined application of the recurrence relations (4.3) and (4.4) for the functions $A_{mn}^{(\nu)}(h, k)$ and the formulas (5.2) for expressing the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$ in terms of the functions $A_{mn}^{(\nu)}(h, k)$. For actual evaluation of the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$ no general solutions free from integral signs or numerical procedures appear to be known, the formulas (5.2), the Sternberg and Kaufman approximation process [7, 8, 9] and Feuerstein's generalized Weber-Schafheitlin integral representations [12] being the most general and useful points of departure for numerical work.

Consider next cases (a) and (∞) for integral values of ν and for similar reasons as previously, consider particularly, the basic set of functions $A_{mnl}(h, k_1, k_2) = A_{mnl}^{(1)}(h, k_1, k_2)$. Clearly again it is not possible even here to give a complete solution free from integral signs or numerical procedures in case (a). However, in the subcase of case (a) in which $|h| + k_1 + k_2 \leq 1$, using methods similar to those used before of substitution of the series (4.6) to (4.11) and so on for the functions $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ into the formulas (5.2) for the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$, we may again obtain a solution in multiple power series; in particular, for the case of half-wave rectification in which the bias $h = 0$ we have double power series expansions of the forms

$$\frac{1}{2} A_{000}(0, k_1, k_2) = \frac{1}{\pi} \left\{ \begin{array}{l} 1 + \frac{1}{4} k_1^2 + \frac{1}{64} k_1^4 + \frac{1}{256} k_1^6 + \cdots \\ + \frac{1}{4} k_2^2 + \frac{1}{16} k_1^2 k_2^2 + \frac{9}{256} k_1^4 k_2^2 + \cdots \\ + \frac{1}{64} k_2^4 + \frac{9}{256} k_1^2 k_2^4 + \cdots \\ + \frac{1}{256} k_2^6 + \cdots \\ \cdots \end{array} \right\}, \tag{5.7}$$

$$A_{100}(0, k_1, k_2) = \frac{1}{2}, \qquad A_{010}(0, k_1, k_2) = \frac{1}{2} k_1, \qquad A_{001}(0, k_1, k_2) = \frac{1}{2} k_2, \tag{5.8}$$

$$A_{200}(0, k_1, k_2) = \frac{1}{\pi} \left\{ \begin{array}{l} \frac{2}{3} - \frac{1}{2} k_1^2 + \frac{3}{32} k_1^4 + \frac{5}{384} k_1^6 + \cdots \\ - \frac{1}{2} k_2^2 + \frac{3}{8} k_1^2 k_2^2 + \frac{15}{128} k_1^4 k_2^2 + \cdots \\ + \frac{3}{32} k_2^4 + \frac{15}{128} k_1^2 k_2^4 + \cdots \\ + \frac{5}{384} k_2^6 + \cdots \\ \cdots \end{array} \right\}, \tag{5.9}$$

$$A_{110}(0, k_1, k_2) = \frac{k_1}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{8}k_1^2 - \frac{1}{64}k_1^4 - \frac{5}{1024}k_1^6 - \cdots \\ -\frac{1}{4}k_2^2 - \frac{3}{32}k_1^2k_2^2 - \frac{15}{256}k_1^4k_2^2 - \cdots \\ -\frac{3}{64}k_2^4 - \frac{45}{512}k_1^2k_2^4 - \cdots \\ -\frac{5}{256}k_2^6 - \cdots \\ \cdots \end{array}\right\}, \tag{5.10}$$

$$A_{101}(0, k_1, k_2) = \frac{k_2}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{4}k_1^2 - \frac{3}{64}k_1^4 - \frac{5}{256}k_1^6 - \cdots \\ -\frac{1}{8}k_2^2 - \frac{3}{32}k_1^2k_2^2 - \frac{45}{512}k_1^4k_2^2 - \cdots \\ -\frac{1}{64}k_2^4 - \frac{15}{256}k_1^2k_2^4 - \cdots \\ -\frac{5}{1024}k_2^6 - \cdots \\ \cdots \end{array}\right\}, \tag{5.11}$$

$$A_{020}(0, k_1, k_2) = \frac{k_1^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{48}k_1^2 + \frac{3}{512}k_1^4 + \frac{5}{2048}k_1^6 + \cdots \\ +\frac{1}{16}k_2^2 + \frac{3}{64}k_1^2k_2^2 + \frac{75}{2048}k_1^4k_2^2 + \cdots \\ +\frac{9}{256}k_2^4 + \frac{75}{1024}k_1^2k_2^4 + \cdots \\ +\frac{25}{1024}k_2^6 + \cdots \\ \cdots \end{array}\right\}, \tag{5.12}$$

$$A_{011}(0, k_1, k_2) = \frac{k_1k_2}{\pi}\left\{\begin{array}{l} \frac{1}{2} + \frac{1}{16}k_1^2 + \frac{3}{128}k_1^4 + \frac{25}{2048}k_1^6 + \cdots \\ +\frac{1}{16}k_2^2 + \frac{9}{128}k_1^2k_2^2 + \frac{75}{1024}k_1^4k_2^2 + \cdots \\ +\frac{3}{128}k_2^4 + \frac{75}{1024}k_1^2k_2^4 + \cdots \\ +\frac{25}{2048}k_2^6 + \cdots \\ \cdots \end{array}\right\}, \tag{5.13}$$

$$A_{002}(0, k_1, k_2) = \frac{k_2^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{16}k_1^2 + \frac{9}{256}k_1^4 + \frac{25}{1024}k_1^6 + \cdots \\ +\frac{1}{48}k_2^2 + \frac{3}{64}k_1^2k_2^2 + \frac{75}{1024}k_1^4k_2^2 + \cdots \\ +\frac{3}{512}k_2^4 + \frac{75}{2048}k_1^2k_2^4 + \cdots \\ +\frac{5}{2048}k_2^6 + \cdots \\ \cdots \end{array}\right\}, \tag{5.14}$$

and so forth, each of which converges uniformly for $k_1 + k_2 \leq 1$, the k_1 and k_2 being positive and, in fact as before, the convergence is quite rapid, the terms given above, for example, being sufficient to evaluate the functions $\frac{1}{2}A_{000}(0, k_1, k_2)$ to $A_{002}(0, k_1, k_2)$ to better than 1×10^{-3} units for $k_1 + k_2 \leq \frac{1}{2}$. In case (∞) as before we have

$$\begin{gathered} \tfrac{1}{2}A_{000}(h, k_1, k_2) = |h|, \qquad A_{100}(h, k_1, k_2) = 1, \qquad A_{010}(h, k_1, k_2) = k_1, \\ A_{001}(h, k_1, k_2) = k_2, \qquad A_{mnl}(h, k_1, k_2) = 0, \end{gathered} \tag{5.15}$$

where $m + n + l \geq 2$. As before for half-integral values of ν in the sub-case of case (a) in which $|h| + k_1 + k_2 \leq 1$ comparable double or triple power series expansions for the functions $A_{mnl}^{(\nu)}(0, k_1, k_2)$ or $A_{mnl}^{(\nu)}(h, k_1, k_2)$ also may be obtained by combined use of the formulas (3.4) for the functions $A_m^{(\nu)}(h)$, formulas (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$, and formulas (5.2) for the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$, the half-integral character of ν again making possible the explicit computation of the coefficients in the series but in case (∞) no special results exist. Finally, in addition to the above formulas and series for the functions $A_{mnl}(h, k_1, k_2) = A_{mnl}^{(1)}(h, k_1, k_2)$ in cases (a) and (∞) we may note the reflection relations

$$\begin{aligned} \tfrac{1}{2}A_{000}(-h, k_1, k_2) &\equiv \tfrac{1}{2}A_{000}(h, k_1, k_2) + h, \\ A_{100}(-h, k_1, k_2) &\equiv 1 - A_{100}(h, k_1, k_2), \\ A_{010}(-h, k_1, k_2) &\equiv k_1 - A_{010}(h, k_1, k_2), \\ A_{001}(-h, k_1, k_2) &\equiv k_2 - A_{001}(h, k_1, k_2), \\ A_{mnl}(-h, k_1, k_2) &\equiv (-1)^{m+n+l}A_{mnl}(h, k_1, k_2), \end{aligned} \tag{5.16}$$

for determining the functions of negative bias in terms of those of positive bias where again $m + n + l \geq 2$. These formulas likewise may be established in an elementary manner by the use of previous results for the functions $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ and the formulas (5.2) for the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$.

6. Applications of the theory to the computation of power. The theory of the preceding sections concerning the functions $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ and the graphs of the functions $A_m(h) = A_m^{(1)}(h)$ and $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ given in this paper not only may be applied to the computation of modulation product amplitudes but also to the computation of the average output power of the rectifier (2.1) when responding to a several-frequency input of the form (2.2), (2.3), or (2.4), provided, in the two- and three-frequency problems, that the input is non-periodic. Thus, applying

the Parseval theorem [15, 16] or Bessel equality for simple or multiple Fourier series and assuming for the moment that the average output power $\mathcal{P}_0$ in each case may be taken to be one-half of the sum of the squares of the corresponding Fourier coefficients, we have in the one-, two-, and three-frequency problems respectively the formulas

$$\mathcal{P}_0 = \frac{1}{4} P^{2\nu} A_0^{(\nu)2}(h) + \frac{1}{2} P^{2\nu} \sum_{m=1}^{\infty} A_m^{(\nu)2}(h) = \frac{1}{2} P^{2\nu} A_0^{(2\nu)}(h), \tag{6.1}$$

$$\mathcal{P}_0 = \frac{1}{4} P^{2\nu} A_{00}^{(\nu)2}(h, k) + \frac{1}{2} P^{2\nu} \sum_{m,n=0}^{\infty}{}^{*} A_{mn}^{(\nu)2}(h, k) = \frac{1}{2} P^{2\nu} A_{00}^{(2\nu)}(h, k), \tag{6.2}$$

and

$$\begin{aligned} \mathcal{P}_0 = \frac{1}{4} P^{2\nu} A_{000}^{(\nu)2}(h, k_1, k_2) \\ + \frac{1}{2} P^{2\nu} \sum_{m,n,l=0}^{\infty}{}^{*} A_{mnl}^{(\nu)2}(h, k_1, k_2) = \frac{1}{2} P^{2\nu} A_{000}^{(2\nu)}(h, k_1, k_2), \end{aligned} \tag{6.3}$$

provided, for the validity of (6.2), that the input frequency ratio p/q in (2.3) is irrational and, for the validity of (6.3), that the input frequency ratios p/q, q/r, and r/p in (2.4) are all irrational or, in short, that the inputs in question are non-periodic. The proofs of these formulas follow at once from the theorems cited while the justification of the assumptions made concerning the definition of average output power will be given in the last section below. We note in passing that by considering separately the leading terms and the summations in the left hand members of (6.1), (6.2), and (6.3) we may separate out the alternating and direct current output power from the average output power without difficulty. Finally, it may be remarked that by applying the Sternberg and Kaufman approximation process [7, 8, 9] to the squared characteristics of more general continuous modulators than rectifiers, output power computations for these more general problems may also be carried out in terms of the same Bennett functions as here.

7. Some concluding remarks. In concluding this paper on multiple harmonic analysis in rectifier problems it seems worthwhile to note that the simple and multiple Fourier series expansions of the rectifier output (2.5), (2.6), and (2.7) are also generalized Fourier series in the sense of Bohr [18, 19, 20] provided, in the two- and three-frequency problems, that the inputs are non-periodic and that $\nu > 0$. The significance of this remark and also its justification lies in the fact that under the conditions noted we have for the functions $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ the further formulas

$$A_m^{(\nu)}(h) = 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T Y^\nu[\cos(pt + \theta_p); h] \cos(\omega_m t + \phi_m)\, dt, \tag{7.1}$$

$$\begin{aligned} A_{mn}^{(\nu)}(h, k) = 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T Y^\nu[\cos(pt + \theta_p) \\ + k \cos(qt + \theta_q); h] \cos(\omega_{mn} t + \phi_{mn})\, dt, \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} A_{mnl}^{(\nu)}(h, k_1, k_2) = 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T Y^\nu[\cos(pt + \theta_p) \\ + k_1 \cos(qt + \theta_q) + k_2 \cos(rt + \theta_r); h] \cos(\omega_{mnl} t + \phi_{mnl})\, dt, \end{aligned} \tag{7.3}$$

where for the validity of (7.2) and (7.3) we thus assume that the frequency ratio p/q in (7.2) and the frequency ratios p/q, q/r, and r/p in (7.3) are all irrational and that $\nu > 0$. To establish these formulas we begin by noting with the aid of Hobson [15] and Tonelli [16] that each of the corresponding Fourier series expansions (2.5), (2.6), and (2.7) converges uniformly for $\nu > 0$. To complete the proof of (7.2), for example, we now multiply (2.6) by $\cos(\omega_{\pm m'n'}t + \phi_{\pm m'n'})$ and note that the resulting series also converges uniformly for $\nu > 0$ so that for each $\epsilon > 0$ there exist integers $M > m'$ and $N > n'$ such that the remainder of the new series after the terms

$$P^\nu A^{(\nu)}_{\pm MN}(h, k) \cos(\omega_{\pm MN}t + \phi_{\pm MN}) \cos(\omega_{\pm m'n'}t + \phi_{\pm m'n'}), \tag{7.4}$$

never exceeds $\frac{1}{2}\epsilon P^\nu$ in absolute value regardless of the value of t. Consequently, integrating termwise up to and including the terms (7.4), multiplying through the resulting relation by $2/TP^\nu$, and taking limits we may thence interchange the order of limiting and summation operations in this finite part of the series to conclude that

$$\begin{aligned} \Bigg| A^{(\nu)}_{\pm m'n'}(h, k) - 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T Y^\nu[\cos(pt + \theta_p) \\ + k \cos(qt + \theta_q); h] \cos(\omega_{\pm m'n'}t + \phi_{\pm m'n'})\, dt \Bigg| \\ \leq \frac{2}{P^\nu} \lim_{T\to\infty} \frac{1}{T} \int_0^T \tfrac{1}{2}\epsilon P^\nu\, dt = \epsilon, \end{aligned} \tag{7.5}$$

all of which operations are permissible and valid provided the ratio p/q is irrational and $\nu > 0$ so that (7.2) is thereby proven. Similar proofs may be given of course for formulas (7.1) and (7.3) also.

Consider now the implications of the formulas (7.1), (7.2), and (7.3). First of all, these formulas applied to the constant terms in the various Fourier series expansions considered, assert that under the stated conditions time averages are equal to phase averages. Secondly, since squaring $Y^\nu(X; X_0)$ is equivalent to replacing ν by 2ν, it is now clear that the formulas (6.1), (6.2), and (6.3) previously given under the same conditions for the average output power $\mathcal{P}_0$ were indeed correct, at least for $\nu > 0$. In closing we may note, finally, that in a sense we have here a simplified example of an ergodic situation while the input function and output series are always almost periodic functions in the sense of Bohr.

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