valid for $|\delta| < 1$, or, finally,

$$\Lambda(\epsilon) = \sum_{n=0}^{\infty} \lambda_n [\epsilon/(k + \epsilon)]^n, \tag{4}$$

valid for $\epsilon \geq 0$, and, actually, for $Re(\epsilon) > -k$.

3. The expansion for the associated characteristic function.
Consider the solution of

$$u'' + \Lambda(\epsilon)[f(x) + \epsilon g(x)]u = 0,$$
$$u(0) = 0, \qquad u'(0) = 1. \tag{1}$$

The classical iteration procedure for obtaining u shows that u is an analytic function of $\Lambda(\epsilon)$ and ϵ for $0 \leq x \leq 1$, for all ϵ and $\Lambda(\epsilon)$. Consequently, u is an analytic function of ϵ for $Re(\epsilon) > b$, uniformly in $0 \leq x \leq 1$, and thus possesses an expansion

$$u = \sum_{n=0}^{\infty} u_n(x)[\epsilon/(k + \epsilon)]^n, \tag{2}$$

valid for $\epsilon \geq 0$.

4. Perturbation procedure.
To determine the sequences $\{\lambda_n\}$ and $\{u_n(x)\}$, we employ a standard perturbation technique starting with the equation

$$u'' + \lambda[f(x) + k\delta g(x)/(1 - \delta)]u = 0,$$
$$u(0) = u(1) = 0. \tag{1}$$

References

1. R. Bellman, *On perturbation methods involving expansions in terms of a parameter*, Quart. Appl. Math. **XIII**, 195-200 (1955)
2. R. Bellman, *Research Problem No. 1*, Bull. Am. Math. Soc. **63**, No. 1, 59 (1957)
3. T. Fort, *On a problem of Richard Bellman*, Proc. Am. Math. Soc., to appear

THE NUMERICAL SOLUTION OF AN INFINITE SET OF LINEAR SIMULTANEOUS EQUATIONS*

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The usual method for numerical solution of

$$\sum_{s=1}^{\infty} a_{rs}x_s = b_r, \qquad (r = 1, 2, \cdots), \tag{1}$$

is to solve the first n equations in n unknowns for some fixed n, and to regard the results as approximations to the x_s. The accuracy of this approximation is invariably doubtful. A slight modification of the Choleski[1] or Crout[2] method for solving a set of simultaneous linear equations can be used to provide a systematic procedure for solving the first $m \times m$ equations with $m = 1, 2, 3, \cdots$ in succession. It is then possible to estimate in various ways the error involved in solving only a finite number of equations. The following discussion should be intelligible from either the Crout or the Choleski point of view.

Using matrix notation, the equations $\mathbf{AX} = \mathbf{B}$ are solved by decomposing the matrix $\mathbf{A}$ into a product of triangular matrices $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix. The following systems are then solved in succession:

$$\mathbf{LY} = \mathbf{B}: \quad \mathbf{UX} = \mathbf{Y}.$$

We shall adopt the layout used by Crout in which we start with an "original matrix" $\mathbf{D}$ and write down an "auxiliary matrix" $\mathbf{E}$:

$$\mathbf{D} = \begin{bmatrix} a_{11} , a_{12} , \cdots , a_{1n} , b_1 \\ a_{21} , a_{22} , \cdots , a_{2n} , b_2 \\ \vdots \quad \vdots \qquad \vdots \quad \vdots \\ a_{n1} , a_{n2} , \cdots , a_{nn} , b_n \end{bmatrix} : \quad \mathbf{E} = \begin{bmatrix} l_{11} , u_{12} , \cdots , u_{1n} , y_1 \\ l_{21} , l_{22} , \cdots , u_{2n} , y_2 \\ \vdots \quad \vdots \qquad \vdots \quad \vdots \\ l_{n1} \quad l_{n2} \quad \cdots , l_{nn} , y_n \end{bmatrix},$$

where the l_{rs} , u_{rs} , y_s , are elements of $\mathbf{L}$, $\mathbf{U}$, $\mathbf{Y}$, respectively. For instance $\mathbf{U}$ consists of zeros below the principal diagonal, units in the diagonal and the u_{rs} above the diagonal.

Elements of the auxiliary matrix are determined in the following order:

(i) Elements in the rth column on and below the principal diagonal (i.e. $l_{r,r}$, $l_{r+1,r}$, $\cdots$, $l_{n,r}$).

(ii) Elements in the rth row to the right of the principal diagonal (i.e. $u_{r,r+1}$, $\cdots$, $u_{r,n}$, y_r).

This sequence is performed for $r = 1$, then for $r = 2$, and so on, up to $r = n$, by means of the formulae

$$l_{rs} = a_{rs} - \sum_{m=1}^{s-1} l_{rm} u_{ms} , \quad (r \geq s): \quad u_{rs} = \frac{1}{l_{rr}} \left[a_{rs} - \sum_{m=1}^{r-1} l_{rm} u_{ms} \right], \quad (r < s).$$

The proposed method replaces the solution of $\mathbf{UX} = \mathbf{Y}$ by the following procedure ($x_r^{(m)}$ is the mth approximation to x_r , etc.).

(a) Take as a first approximation to the solution of the infinite system:

$$x_1^{(1)} = y_1 : \quad x_r^{(1)} = 0, \quad (r = 2, 3, \cdots).$$

(b) Obtain a second approximation by solving:

$$z_1^{(2)} + u_{12} z_2^{(2)} = 0$$
$$z_2^{(2)} = y_2 .$$

Then

$$x_1^{(2)} = x_1^{(1)} + z_1^{(2)} : \quad x_2^{(2)} = y_2 : \quad x_r^{(2)} = 0, \quad (r = 3, 4, \cdots).$$

(c) Obtain a third approximation by solving:

$$z_1^{(3)} + u_{12} z_2^{(3)} + u_{13} z_3^{(3)} = 0$$
$$z_2^{(3)} + u_{23} z_3^{(3)} = 0$$
$$z_3^{(3)} = y_3 .$$

Then

$$x_1^{(3)} = x_1^{(2)} + z_1^{(3)} : \quad x_2^{(3)} = x_2^{(2)} + z_2^{(3)} :$$
$$x_3^{(3)} = y_3 : \quad x_r^{(3)} = 0, \quad (r = 4, 5, 6, \cdots) \quad \text{and so on.}$$

The theory behind the method is obvious if we reflect that at the mth stage we have obtained the exact solution of the first m equations in m unknowns.

As an example consider the following original matrix which is derived from the first four equations of an infinite system[3] (the elements in the check column are just the sums of the elements in the corresponding rows of the original matrix):

x_1	x_2	x_3	x_4	Constant	Check
+5.665,118	−0.240,000	+0.059,172	−0.022,400	+2.670,644	8.132,534
−0.240,000	+1.270,836	−0.103,806	+0.049,941	−0.049,299	0.927,672
+0.059,172	−0.103,806	+0.708,321	−0.051,132	+0.006,400	0.618,955
−0.022,400	+0.049,941	−0.051,132	+0.489,615	−0.001,666	0.464,358.

The auxiliary matrix is:

+5.665,118	−0.042,365	+0.010,445	−0.003,954	+0.471,419	1.435,545
−0.240,000	+1.260,668	−0.080,353	+0.038,862	+0.050,641	1.009,150
+0.059,172	−0.101,299	+0.699,563	−0.067,129	−0.023,393	0.909,478
−0.022,400	+0.048,992	−0.046,961	+0.484,470	+0.010,969	1.010,969.

The procedure described above then gives

$$x_1^{(1)} = +0.471,419.$$

$$z_1^{(2)} = +0.002,145: \quad x_2^{(2)} = +0.050,641.$$

$$z_1^{(3)} = +0.000,165: \quad z_2^{(3)} = -0.001,880: \quad x_3^{(3)} = -0.023,393.$$

$$z_1^{(4)} = +0.000,020: \quad z_2^{(4)} = -0.000,367: \quad z_3^{(4)} = +0.000,736: \quad x_4^{(4)} = +0.010,969.$$

The values for $x_s^{(4)}$ agree with those given in[3] to a unit in the sixth decimal place. In addition, an estimate of accuracy can be obtained by comparing the magnitudes of the corrections for successive values of $m = 1, 2, 3, \cdots$. By inspection of the above results it appears that the approximate x_s obtained by solving the 4×4 set of equations are accurate to roughly 5, 4, 3, 3 decimal places for $s = 1, 2, 3, 4$ respectively.

When the corrections $z_s^{(m)}$ do not converge rapidly to zero it will be desirable to give definite upper and lower limits for the unknowns. This may be easy if the corrections alternate in sign. In more difficult cases the following procedure has proved useful. Suppose that we have found the estimates $x_s^{(m)}$ of x_s by solving the first m equations in m unknowns. Suppose that from these or otherwise it is possible to estimate the x_s for $s = m + 1, m + 2, \cdots$. Denote these estimates by $X_s^{(m)}(s > m)$. Then a second estimate of the x_s is given by the quantities $(x_s^{(m)} + u_s^{(m)})$ where the $u_s^{(m)}$ satisfy

$$\sum_{s=1}^{m} a_{rs} u_s^{(m)} = - \sum_{s=m+1}^{\infty} a_{rs} X_s^{(m)}, \qquad (r = 1 \text{ to } m). \tag{2}$$

For sufficiently large m the $u_s^{(m)}$ are small corrections and a comparatively crude estimate of the sum on the right hand side of this equation is adequate. If convenient we can

[3]A. Weinstein and D. H. Rock, Quart. Appl. Math. 2, 262-266 (1944).

replace the a_{rs} by an approximation with the correct behavior for large s. It will often be possible to replace the summation by integration.

As an example consider the following equations[4]:

$$2.759x_1 + 0.355x_2 + 0.302x_3 + 0.267x_4 = 1.0000$$

$$0.214x_1 + 4.457x_2 + 0.227x_3 + 0.214x_4 = 0.3333$$

$$0.143x_1 + 0.178x_2 + 5.691x_3 + 0.178x_4 = 0.2000$$

$$0.107x_1 + 0.142x_2 + 0.151x_3 + 6.703x_4 = 0.1429.$$

These are the first four equations in four unknowns from an infinite system for which $b_r = (2r - 1)^{-1}$ and it is known that for large r,

$$a_{rr} \approx (2\pi)^{1/2}(2r - 1)^{1/2}: \qquad a_{rs} \approx (2s - 1)^{1/2}/\{(2\pi)^{1/2}(r + s - 1)\}. \tag{3}$$

Also all unknowns are positive so that by inspection

$$x_s < a_{ss}^{-1}b_s \approx (2\pi)^{-1/2}(2s - 1)^{-3/2}. \tag{4}$$

We shall use the following estimate in (2):

$$X_s^{(m)} \approx x_m^{(m)}(2m - 1)^{3/2}(2s - 1)^{-3/2}. \tag{5}$$

This will give an overestimate of the magnitude of the right hand side of (2) since from (4) the x_s decrease at least as rapidly as $(2s - 1)^{-3/2}$ and from the numerical results given below it will appear that the $x_m^{(m)}$ are overestimates of the x_m . We replace the right hand side of (2) by

$$-x_m^{(m)} \frac{(2m - 1)^{3/2}}{(2\pi)^{1/2}} \int_{m+1/2}^{\infty} \frac{dx}{(2x - 1)(r + x - 1)}$$

$$= -x_m^{(m)} \frac{(2m - 1)^{3/2}}{(2\pi)^{1/2}(2r - 1)} \log\left(\frac{2r + 2m - 1}{2m}\right).$$

The final results are presented in the following table where the upper figures give $x_s^{(m)}$, the lower give $x_s^{(m)} + u_s^{(m)}$. The convergence of the results from above and below as m is increased indicates that upper and lower limits have been established for the unknowns. But the convergence is much slower than in the example given above and it would be difficult to establish accurate estimates of the unknowns from the upper limits alone.

$$x_1^{(1)} = \begin{matrix} 0.3624 \\ 0.3412 \end{matrix}$$

$$x_1^{(2)} = \begin{matrix} 0.3550 \\ 0.3459 \end{matrix} \qquad x_2^{(2)} = \begin{matrix} 0.0577 \\ 0.0531 \end{matrix}$$

$$x_1^{(3)} = \begin{matrix} 0.3524 \\ 0.3469 \end{matrix} \qquad x_2^{(3)} = \begin{matrix} 0.0566 \\ 0.0536 \end{matrix} \qquad x_3^{(3)} = \begin{matrix} 0.0246 \\ 0.0225 \end{matrix}$$

$$x_1^{(4)} = \begin{matrix} 0.3512 \\ 0.3474 \end{matrix} \qquad x_2^{(4)} = \begin{matrix} 0.0560 \\ 0.0539 \end{matrix} \qquad x_3^{(4)} = \begin{matrix} 0.0242 \\ 0.0226 \end{matrix} \qquad x_4^{(4)} = \begin{matrix} 0.0140 \\ 0.0128. \end{matrix}$$

[4]D. S. Jones, Proc. Roy. Soc. 217, 153-175 (1953).

We next consider evaluation of the following infinite series (see[4]):

$$S = \sum_{s=1}^{\infty} k_s x_s \; ,$$

where

$$k_1^{-1} = 2.33, \qquad k_2^{-1} = 4.22, \qquad k_3^{-1} = 5.51, \qquad k_4^{-1} = 6.55,$$

and $k_s^{-1} \sim (2\pi)^{1/2}(2s - 1)^{1/2}$ for large s. If we evaluate

$$S_n = \sum_{s=1}^{n} k_s x_s^{(n)}$$

we obtain the following results:

$$S_1 = 0.155_5 : \qquad S_2 = 0.166_1 : \qquad S_3 = 0.169_1 : \qquad S_4 = 0.170_5 \; .$$

These underestimate S. In order to obtain overestimates of S we evaluate, using the previous approximation (5),

$$S_n' = \sum_{s=1}^{n} k_s x_s^{(n)} + k_n x_n^{(n)}(2n - 1)^2 \sum_{s=n+1}^{\infty} \frac{1}{(2s - 1)^2}$$

$$\approx \sum_{s=1}^{n} k_s x_s^{(n)} + k_n x_n^{(n)}(2n - 1)^2 (4n)^{-1},$$

where we have replaced summation by integration as before. This gives

$$S_1' = 0.194_4 : \qquad S_2' = 0.181_5 : \qquad S_3' = 0.178_5 : \qquad S_4' = 0.176_9 \; .$$

The mean values $\frac{1}{2}(S_n + S_n')$ are remarkably constant. We can say from these results that $0.171 < S < 0.177$ with a probable value of $S = 0.174$.

ON SIMPLE SUBHARMONICS*

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Introduction. In a recent paper [1],† Rosenberg clarified considerably the curious subharmonic phenomena of non-linear oscillations by introducing the concepts of strong subharmonic solutions and of the simple subharmonics. He considered a system with a single degree of freedom, whose mechanical model might be a mass under the action of an elastic force, linear or non-linear, and of a simple harmonic forcing function of frequencey ω. The equation of motion of the system can be put in the form

$$\frac{d^2x}{dt^2} + f(x) = P_0 \cos \omega t, \tag{1}$$

where P_0 is proportional to the amplitude of the external forcing function.

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†Numbers in square brackets refer to the bibliography at the end of the paper.