EIGENOSCILLATIONS OF AN ELASTIC CABLE*

BY

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Summary. The eigenoscillations of a cable, supported at its end points, in a homogeneous gravitational field is investigated under the assumption that the bending stiffness is negligible but the behavior with respect to tension stresses is perfectly elastic. The problem involves two coupled second order equations and one independent second order equation; it is shown to be definitely self-adjoint and an iterative method for its solution is suggested.

In the particular case of a shallow cable, that is with negligible sag, the asymptotic eigenvalues are obtained. It turns out that the gravitational field has such a "stiffening" effect that the eigenvalues related to some oscillation modes may be substantially greater (the lowest of them eightfold) than those given in the classical theory on vibrating strings.

Notations.

A = cross sectional area
L = total length
s = arc length parameter
ρ = mass density
μ = ρA = mass per unit length
} of the cable when unstrained;

E = Young's modulus;

x, y, z rectilinear coordinates of a typical cable point; the gravity acting in the direction of negative y-axis, the end points of the cable having coordinates $(\pm \frac{1}{2}l, 0, 0)$;

u, v, w displacement components from equilibrium position;

F = tension force acting on the cable;

f = increment of the force due to the motion;

a = length constant determined by Equation (3);

ϵ = $a\mu g/EA = a\rho g/E$ equilibrium strain at the lowest point;

α = $L/2a$ = characteristic parameter;

ω = circular frequency of eigenoscillations;

$\lambda^* = \omega^2 a/g$
$\lambda = \kappa^2 = \omega^2 L^2/4ag$
} eigenvalue parameters;

$p = s/a$, $q = 2s/L$ dimensionless variables;

A, **B**, **M**, **N**, **S**, **T** matrices.

Derivation of the equations. From Hooke's law follows

$$x_s'^2 + y_s'^2 + z_s'^2 = (1 + F/EA)^2. \tag{1}$$

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Hence the components of the tension force are

$$\frac{Fx_s'}{1 + F/EA}, \quad \frac{Fy_s'}{1 + F/EA}, \quad \frac{Fz_s'}{1 + F/EA}.$$

The influence of the flexural rigidity being neglected, Newton's law gives

$$\begin{cases} \left(\dfrac{Fx_s'}{1 + F/EA}\right)_s' = \mu x_{tt}'', \\ \left(\dfrac{Fy_s'}{1 + F/EA}\right)_s' = \mu(y_{tt}'' + g), \\ \left(\dfrac{Fz_s'}{1 + F/EA}\right)_s' = \mu z_{tt}''. \end{cases} \tag{2}$$

If time derivatives are taken to be identically zero, then Eqs. (1) and (2) give rise to the stationary equilibrium solution. Using boundary conditions and choosing the origin of s at the lowest point one obtains

$$\begin{cases} x_0 = a \log [s/a + (1 + s^2/a^2)^{1/2}] + \epsilon s, \\ y_0 = (a^2 + s^2)^{1/2} + \epsilon s^2/2a + K, \\ z_0 = 0 \\ F_0 = \mu g(a^2 + s^2)^{1/2}. \end{cases}$$

The integration constants a and K in these expressions are determined by the conditions

$$\log [L/2a + (1 + L^2/4a^2)^{1/2}] + L\rho g/2E = l/2a, \tag{3}$$

$$(1 + L^2/4a^2)^{1/2} + L^2\rho g/8aE + K/a = 0.$$

After introducing the small oscillation displacements by

$$\begin{cases} x(s, t) = x_0(s) + u(s, t) \\ y(s, t) = y_0(s) + v(s, t) \\ z(s, t) = z_0(s) + w(s, t) \\ F(s, t) = F_0(s) + f(s, t) \end{cases}$$

and substituting these expressions in (1) and (2) the linearization is performed:

$$\frac{u_s' + sv_s'/a}{(1 + s^2/a^2)^{1/2}} = \frac{f}{EA}$$

$$\left(\frac{f + \epsilon EA(1 + s^2/a^2)u_s'}{(1 + s^2/a^2)^{1/2}[1 + \epsilon(1 + s^2/a^2)^{1/2}]}\right)_s' = \mu u_{tt}'',$$

$$\left(\frac{sf/a + \epsilon EA(1 + s^2/a^2)v_s'}{(1 + s^2/a^2)^{1/2}[1 + \epsilon(1 + s^2/a^2)^{1/2}]}\right)_s' = \mu v_{tt}'',$$

$$\left(\frac{\epsilon EA(1 + s^2/a^2)w_s'}{(1 + s^2/a^2)^{1/2}[1 + \epsilon(1 + s^2/a^2)^{1/2}]}\right)_s' = \mu w_{tt}''.$$

When f is eliminated, the dimensionless variable $p = s/a$ introduced, and the variables separated by $u(s, t) = u(s) \cdot e^{i\omega t}$, etc., then the following system of ordinary differential equations is obtained:

$$\begin{cases} \left(\dfrac{u' + pv' + \epsilon(1 + p^2)^{3/2}u'}{\epsilon(1 + p^2)[1 + \epsilon(1 + p^2)^{1/2}]}\right)' + \lambda^* u = 0, \\ \left(\dfrac{pu' + p^2v' + \epsilon(1 + p^2)^{3/2}v'}{\epsilon(1 + p^2)[1 + \epsilon(1 + p^2)^{1/2}]}\right)' + \lambda^* v = 0, \end{cases} \tag{4}$$

$$\left(\frac{w'}{(1 + p^2)^{-1/2} + \epsilon}\right)' + \lambda^* w = 0. \tag{5}$$

Two first equations apparently determine the modes and frequencies of the eigen-oscillations in the xy-plane, whereas the third equation applies to the perpendicular oscillations, completely independent of the first ones. In the following the main attention will be given to the first oscillations.

The self-adjointness of the problem. Since the eigenvalue problem, consisting of the two second order differential equations (4) and boundary conditions

$$u(\pm\alpha) = v(\pm\alpha) = 0, \qquad (\alpha = L/2a)$$

may be regarded as arising from a regular variational problem, it is self-adjoint. The exact definition of a definitely self-adjoint problem, due to Bliss [1, 2], requires that if the problem is written in the form

$$\begin{cases} \dfrac{d}{dp}\mathbf{u} = \mathbf{A}\mathbf{u} + \lambda\mathbf{B}\mathbf{u}, \\ \mathbf{M}\mathbf{u}(a) + \mathbf{N}\mathbf{u}(b) = 0, \end{cases}$$

where $\mathbf{u}$ is an n-vector, $\mathbf{A}$ and $\mathbf{B}$ are $(n \times n)$ matrices and $\mathbf{M}$ and $\mathbf{N}$ constant $(n \times n)$ matrices with $(\mathbf{M}, \mathbf{N})$ being of rank n, then there exists a non-singular matrix $\mathbf{T}$ such that the following conditions are fulfilled:

$$\begin{cases} \dfrac{d}{dp}\mathbf{T} + \mathbf{T}\mathbf{A} + \mathbf{A}'\mathbf{T} = 0, \\ \mathbf{T}\mathbf{B} + \mathbf{B}'\mathbf{T} = 0, \end{cases} \tag{6}$$

$$\mathbf{M}\mathbf{T}^{-1}(a)\mathbf{M}' = \mathbf{N}\mathbf{T}^{-1}(b)\mathbf{N}', \tag{7}$$

(here the prime stands for the transposition of the matrix in question) and that the matrix

$$\mathbf{S} = \mathbf{T}'\mathbf{B}$$

is symmetric and definite or semi-definite.

In the present case, solve the Eqs. (4) with respect to u'' and v'' and regard u, v, u', and v' as four elements of $\mathbf{u}$. One finds then that the coefficient matrices are

$$\mathbf{A} = \frac{1}{\epsilon(1 + p^2)^{5/2}[1 + \epsilon(1 + p^2)^{1/2}]} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p + \epsilon p(2 - p^2)(1 + p^2)^{1/2} & p^2 - \epsilon(1 - 2p^2)(1 + p^2)^{1/2} \\ 0 & 0 & -1 - \epsilon(1 - 2p^2)(1 + p^2)^{1/2} & -p - 3\epsilon p(1 + p^2)^{1/2} \end{bmatrix},$$

$$\mathbf{B} = \frac{1}{(1+p^2)^{3/2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p^2 - \epsilon(1+p^2)^{3/2} & p & 0 & 0 \\ p & -1 - \epsilon(1+p^2)^{3/2} & 0 & 0 \end{bmatrix},$$

$$(\mathbf{M}, \mathbf{N}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The solution for (6), (7) is now

$$\mathbf{T} = \frac{1}{\epsilon(1+p^2)[1+\epsilon(1+p^2)^{1/2}]}$$

$$\cdot \begin{bmatrix} 0 & 0 & 1+\epsilon(1+p^2)^{3/2} & +p \\ 0 & 0 & +p & p^2+\epsilon(1+p^2)^{3/2} \\ -1-\epsilon(1+p^2)^{3/2} & -p & 0 & 0 \\ -p & -p^2-\epsilon(1+p^2)^{3/2} & 0 & 0 \end{bmatrix}$$

and the matrix $\mathbf{S}$ is semi definite:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The definitely self-adjoint character of the problem thus being proved, a number of properties associated with such problems may be applied for the present problem: for example, the existence of a countably infinite number of eigensolutions, the expansion theorem, etc. The orthogonality of two independent eigensolutions (u_1, v_1), (u_2, v_2) now takes the form

$$\int_{-\alpha}^{+\alpha} (u_1 u_2 + v_1 v_2)\, dp = 0.$$

Furthermore, an iterative method for the determination of the eigensolutions is applicable and proved to be convergent [3]. The sequence of consecutive iterates is obtained directly from (4):

$$\begin{cases} u_{i+1} = \int \left\{ \left(-\dfrac{p^2}{(1+p^2)^{3/2}} - \epsilon \right) \int u_i\, dp + \dfrac{p}{(1+p^2)^{3/2}} \int v_i\, dp \right\} dp, \\ v_{i+1} = \int \left\{ \dfrac{p}{(1+p^2)^{3/2}} \int u_i\, dp + \left(-\dfrac{1}{(1+p^2)^{3/2}} - \epsilon \right) \int v_i\, dp \right\} dp; \end{cases}$$

the integration constants must be chosen to make the functions satisfy the boundary conditions.

Finally, it may be seen directly from the form of the Eq. (5) that the problem for the determination of the lateral oscillations is definitely self-adjoint also and hence subject to similar properties and procedures as those mentioned above.

Case of a shallow cable. The general solution of the problem above depends essentially on two dimensionless parameters only, the relative cable length $\alpha = L/2a$ and the minimum strain ϵ. The latter is always smaller than the ratio between the yield stress and Young's modulus and hence small compared with 1. In the case of a shallow cable the first parameter also is small compared with 1, since a is approximately equal to the smallest radius of curvature. For a closer study of this particular case we replace the variable p by the new variable

$$q = 2s/L = p/\alpha$$

in order to have fixed boundary values $q = \pm 1$. The Eqs. (4) and (5) thus obtain the form

$$\begin{cases} \left(\dfrac{u' + \alpha q v' + \epsilon(1 + \alpha^2 q^2)^{3/2} u'}{\epsilon(1 + \alpha^2 q^2)[1 + \epsilon(1 + \alpha^2 q^2)^{1/2}]}\right)' + \lambda u = 0, \\ \left(\dfrac{\alpha q u' + \alpha^2 q^2 v' + \epsilon(1 + \alpha^2 q^2)^{3/2} v'}{\epsilon(1 + \alpha^2 q^2)[1 + \epsilon(1 + \alpha^2 q^2)^{1/2}]}\right)' + \lambda v = 0, \end{cases} \tag{8}$$

$$\left(\frac{w'}{(1 + \alpha^2 q^2)^{-1/l} + \epsilon}\right)' + \lambda w = 0, \tag{9}$$

where the prime now indicates differentiation with respect to q. The pertinent boundary conditions are

$$u(\pm 1) = v(\pm 1) = w(\pm 1) = 0. \tag{10}$$

After noticing that the last equation apparently has the asymptotic eigenvalues

$$\lambda \sim (n\pi/2)^2, \qquad n = 1, 2, \cdots,$$

for α and ϵ tending to zero, we will leave this equation aside and study the problem (8), (10) for small values of α and ϵ. By letting, separately, $\alpha \to 0$ one arrives at the equations

$$\begin{cases} u'' + \epsilon\lambda u = 0, \\ v'' + (1 + \epsilon)\lambda v = 0, \end{cases}$$

which are not coupled and give the well-known eigenfrequencies of longitudinal and transversal oscillations. Since there is, for a small ϵ, a wide gap between the smallest eigenvalues $\pi^2/4\epsilon$ and $\sim \pi^2/4$ of both groups, it is of interest to find out what are actually the smallest eigenvalues, say when $\epsilon\lambda$ may be regarded to be still essentially smaller than 1. To this end observe, that the Eqs. (8) are equivalent with the following two equations, where C_1 and C_2 are arbitrary integration constants:

$$u' + \alpha q v' + \epsilon\lambda\left[\int_0^q u\, dq + \alpha q \int_0^q v\, dq\right] + \epsilon C_1 + \epsilon C_2 \alpha q = 0, \tag{11}$$

$$\frac{\alpha q u' - v'}{\epsilon + (1 + \alpha^2 q^2)^{-1/2}} + \lambda\left[\alpha q \int_0^q u\, dq - \int_0^q v\, dq\right] + C_1 \alpha q - C_2 = 0. \tag{12}$$

Then, considering first the solutions of group I, that is, those for which u is odd and v is even, one immediately finds that $C_2 = 0$. Assume that λ remains bounded for α and ϵ tending to zero and, since both integral terms of Eq. (11) have ϵ as a factor, neglect these terms besides two first terms:

$$u' + \alpha q v' + \epsilon C_1 = 0. \tag{13}$$

Further, substitute u from this equation into the Eq. (12) and neglect all terms which are not of the lowest order in α or ϵ:

$$v' + \lambda \int_0^q v\, dq - C_1 \alpha q = 0.$$

All even solutions of this equation which satisfy the condition $v(1) = 0$, are, up to a constant factor, of the form

$$v = \cos \kappa q - \cos \kappa,$$

whereby C_1 and λ are

$$C_1 = -\kappa^2 \cos \kappa/\alpha, \qquad \lambda = \kappa^2.$$

The substitution into (13) gives odd solutions

$$u = \alpha[\sin \kappa q/\kappa - q \cos \kappa q + \epsilon \kappa^2 q \cos \kappa/\alpha^2].$$

From the condition $u(1) = 0$, κ is determined as a solution of the characteristic equation

$$\tan \kappa/\kappa = 1 - \kappa^2 \epsilon/\alpha^2.$$

Hence, the roots of this equation squared give the eigenvalues.

Quite similarly one finds that the asymptotic solutions of group II, whose u is even and v odd, are

$$\begin{cases} \lambda = n^2\pi^2, \\ u = -(\alpha/n\pi)(\cos n\pi + \cos n\pi q + n\pi q \sin n\pi q), \\ v = \sin n\pi q. \end{cases}$$

Hence in group I the solutions, the eigenvalues as well as the modes of eigenfunctions, depend on the ratio ϵ/α^2, whereas in group II the solutions are independent of this ratio. The dependence is illustrated in Fig 1, where the solid curves give some lowest eigenvalues from the group I, the broken lines eigenvalues from group II. In particular, for $\alpha^2/\epsilon = n^2\pi^2$ there exist double eigenvalues $n^2\pi^2$.

Finally, it may be indicated what the physical significance of the ratio ϵ/α^2 is. Expressing ϵ as the ratio σ/E, where σ is the equilibrium stress at the lowest point, and α by

$$\alpha = L/2a = L\rho g/2\epsilon E = L\rho g/2\sigma,$$

we obtain

$$\epsilon/\alpha^2 = 4\sigma^3/L^2\rho^2 g^2 E = 4\cdot(L_0/L)^2,$$

where the characteristic length

$$L_0 = (\sigma/E)^{1/2}\cdot(\sigma/\rho g)$$

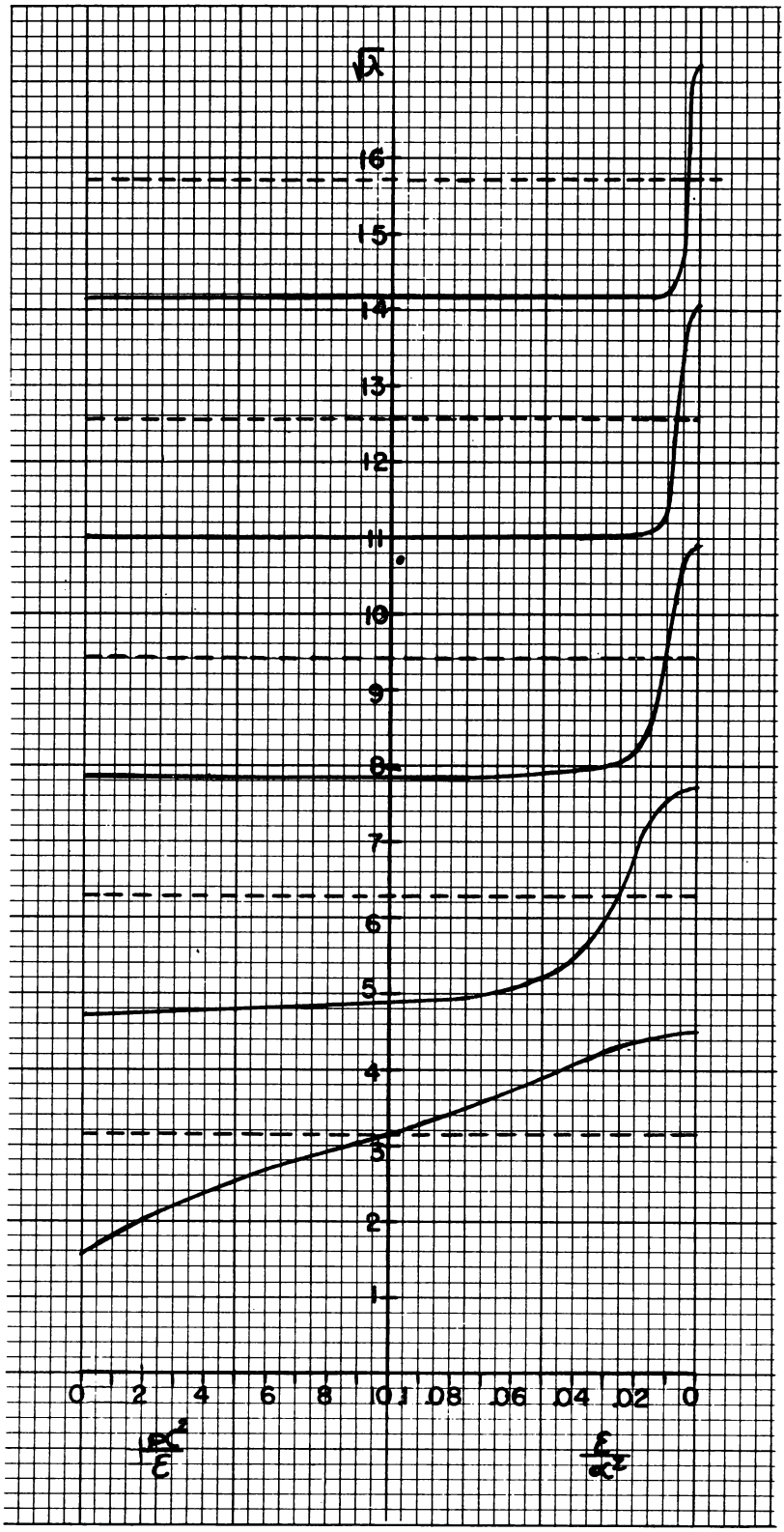
$\sqrt{\lambda}$
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
2
4
6
8
10
.08
.06
.04
.02
0
$\frac{\rho c^2}{E}$
$\frac{E}{\rho c^2}$

FIG. 1

depends on the material in question and its equilibrium stress. For most materials this length, when computed for stresses reasonably close to their tensile proportionality limit, amounts to several hundreds of feet. For instance, for copper, $\sigma = 30{,}000$ psi, $L_0 = 330$ feet, and for steel, $\sigma = 40{,}000$ psi, $L_0 = 430$ feet. Therefore, the eigenfrequencies of a tightly stressed cable or wire do essentially differ from those corresponding to the well-known eigenvalues $\lambda = (n\pi/2)^2$ only at spans of length comparable with this measure.

References

[1] G. A. Bliss, *A boundary value problem for a system of ordinary differential equations of first order*, TAMS **28**, 561-584 (1926)

[2] G. A. Bliss, *Definitely self-adjoint boundary value problems*, TAMS **44**, 413-428 (1938)

[3] P. Laasonen, *Bemerkung zur iterativen Lösung der Eigenwertaufgabe einer Vektordifferentialgleichung*, Ann. Acad. Sci. Fenn. A I **230**, 1-8 (1956)