For the stationary case (25) is identical, except for a change in notation, with Eq. (79) of Ref. [1]. In terms of the matrix \( g(r, \mu, \eta) \) the characteristic function can be written

\[
F(\eta) = \exp \int_0^\eta d\xi \int_0^r \text{Tr}[g(\sigma, \sigma; \xi)h(\sigma)] d\sigma, \tag{26}
\]

where \( \text{Tr}[\ ] \) is the trace of the matrix.

**References**

2. The symmetry of (1) allows \( h(x) \) to be symmetric without loss of generality

**FURTHER EXTENSIONS OF SCHUSTER'S INTEGRAL**

By E. T. KORNHAUSER** (H. H. Wills Physics Laboratory, University of Bristol)

The integral,

\[
I = \int_0^\infty \left[ C(x) + S^2(x) \right] dx,
\]

where \( C(x) \) and \( S(x) \) are Fresnel integrals defined by

\[
C(x) = \int_x^\infty \cos \xi^2 \, d\xi, \\
S(x) = \int_x^\infty \sin \xi^2 \, d\xi,
\]

was conjectured by Schuster\(^1\) to have the value \((\pi/8)^\frac{1}{2}\). Proof that \( I \) does in fact have this value was given by Hardy\(^2\) and more elegantly by Ingham\(^3\). More recently Bateman\(^4\) has extended Ingham's treatment to evaluate integrals of the form

\[
\int_0^\infty C(x)C(ax) \, dx, \quad \int_0^\infty C(x)S(ax) \, dx,
\]

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Consider the integral

\[ I_1 = \int_0^\infty C^2(x + a) \, dx = \int_a^\infty C^2(x) \, dx = [xC^2(x)]_a^\infty + 2 \int_a^\infty x \cos x^2 C(x) \, dx \]

\[ = -aC^2(a) + \left[ \sin x^2 C(x) \right]_a^\infty + \int_a^\infty \sin x^2 \cos x^2 \, dx \]

\[ = -aC^2(a) - \sin a^2 C(a) + \frac{1}{2} 2^{-1/2} S(2^{1/2}a). \]

In like manner

\[ I_2 = \int_0^\infty S^2(x + a) \, dx = -aS^2(a) + \cos a^2 S(a) - \frac{1}{2} 2^{-1/2} S(2^{1/2}a). \]

Furthermore,

\[ I_3 = \int_0^\infty [C^2(x + a) + S^2(x + a)] \, dx = I_1 + I_2 \]

\[ = -a[C^2(a) + S^2(a)] + \cos a^2 S(a) - \sin a^2 C(a). \]

Changing the sign of \(a\) yields

\[ I_4 = \int_0^\infty [C^2(x - a) + S^2(x - a)] \, dx \]

\[ = a[C^2(-a) + S^2(-a)] + \cos a^2 S(-a) - \sin a^2 C(-a), \]

and since \(C(-a) = (\pi/2)^{1/2} - C(a)\) and similarly for \(S(-a)\),

\[ I_3 + I_4 = (\pi/2)^{1/2}(\cos a^2 - \sin a^2) + 2a(\pi/2)^{1/2}[\pi/2]^{1/2} - C(a) - S(a)]. \]

Now consider the integral,

\[ I_5 = \int_0^\infty [C(x + a)C(x - a) + S(x + a)S(x - a)] \, dx. \]

Successive integrations by parts yield

\[ I_5 = \frac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) + a \int_0^\infty [C(x + a) \cos (x - a)^2 \]

\[ - C(x - a) \cos (x + a)^2 + S(x + a) \sin (x - a)^2 - S(x - a) \sin (x + a)^2] \, dx, \]

which may be written

\[ I_5 = \frac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) + a \int_0^\infty \frac{\partial}{\partial a} [C(x + a)C(x - a) + S(x + a)S(x - a)] \, dx \]

Thus it is clear that \(I_5\) obeys the differential equation,

\[ I_5 - a \frac{dI_5}{da} = \frac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) = F(a) \]
which may be solved by quadrature,

\[ I_5 = a \int_a^\infty u^{-2}F(u) \, du = \frac{1}{2}(\pi/2)^{1/2}\{\cos a^2 - \sin a^2 - 2a[\mathcal{C}(a) + S(a)]\}. \]

Finally, the most significant integral from the point of view of the physical problem

\[ I_6 = \int_0^\infty \{(C(x + a) - C(x - a))^2 + (S(x + a) - S(x - a))^2\} \, dx \]

\[ = I_3 + I_4 - 2I_5 = \pi a, \]

a remarkably simple result.

**NOTE ON THE POINCARÉ BOUNDARY-VALUE PROBLEM**

By E. E. JONES (University of Nottingham, England)

1. This note is concerned with the solution of a modified form of the Poincaré boundary-value problem [1]. It is required to solve the Poisson differential equation

\[ \nabla^2 \phi_i = f(x, y) \text{ for } \phi_i(x, y) \text{ defined in } S_i, \]

the region enclosed by the circle \( C \) of equation \(| z | = a, (z = x + iy)\), such that on \( C \)

\[ k \frac{\partial \phi_i}{\partial n} + l \frac{\partial \phi_i}{\partial s} + m \phi_i = g(x, y), \tag{1} \]

where \( k, l, m \) are constants, \( f(x, y) \) is prescribed in \( S_i \), \( g(x, y) \) is prescribed on \( C \), and \( \partial/\partial n, \partial/\partial s \) denote differentiations along the inward normal and positive tangential directions respectively to \( C \).

It is assumed that \( \phi_i = \phi_{i0} + \Phi_i \), where \( \phi_{i0}(x, y) \) is a particular integral of the Poisson equation reflecting all the singularities of the complete solution \( \phi_i \), and \( \Phi_i(x, y) \) is harmonic in \( S_i \), being together with its first partial derivatives single-valued and continuous in \( S_i \). It is thus possible to write \( \Phi_i = re \, W_i(z) \), where \( W_i(z) \) is a regular function of \( z \) in \( S_i \). If \( z = ae^{i\theta} = \zeta, \bar{z} = a^2/\zeta \) on \( C \), and by definition

\[ h(\zeta) = -g - k \frac{\partial \phi_{i0}}{\partial r} + \frac{l}{a} \frac{\partial \phi_{i0}}{\partial \theta} + m \phi_{i0}, \tag{2} \]

then

\[ h_i(z) = \frac{1}{2\pi i} \int_c \frac{(\zeta + z)h(\zeta)}{\zeta - z} \, d\zeta, \tag{3} \]

is the Schwarz integral representation of a function \( h_i(z) \) regular in \( S_i \) with a real part equal to \( h(\zeta) \) on \( C \)—for this to be so it is necessary for \( h(\zeta) \) to satisfy the Lipschitz condition on \( C \) [2]. The boundary condition (1) then takes the form

\[ \Re \left\{ \frac{\xi}{\alpha} \frac{dW_i(\zeta)}{d\zeta} - mW_i(\zeta) - h_i(\zeta) \right\} = 0, \tag{4} \]

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