The question of uniqueness concerns the possibility of two or more solutions to a boundary value problem of the type just described.

Suppose now that two solutions exist. Let \( \Delta \sigma_{ij}(x, t) \), \( \Delta \epsilon_{ij}(x, t) \) and \( \Delta \nu_i \) denote the differences of these solutions. We then have—in view of some of the regularity requirements stated above—from the divergence theorem [5] and the boundary conditions (15)

\[
\int_D \frac{\partial}{\partial x_i} (\Delta \sigma_{ij} \Delta \nu_i) \, dV = \int_B \Delta \sigma_{ij} \Delta \nu_i n_i \, dS = \int_B \Delta T_i \Delta \nu_i \, dS = 0. \tag{16}
\]

On the other hand with the use of (3) and (4), (16) reduces to

\[
\int_D \Delta \sigma_{ij} \Delta \epsilon_{ij} \, dV = 0.
\]

Now by integrating (16) with respect to \( t \) and noting that \( \Delta \sigma_{ij} \) and \( \Delta \epsilon_{ij} \) also satisfy the constitutive law (2) we obtain

\[
\int_0^T dt \int_D \Delta \sigma_{ij} \Delta \epsilon_{ij} \, dV = \int_D \left( \int_0^T \Delta \sigma_{ij} \Delta \epsilon_{ij} \, dt \right) \, dV = \int_D W[x, \Delta \epsilon_{ij}] \, dV = 0. \tag{17}
\]

If \( W \) is positive definite then (17) demands that

\[
W[x, \Delta \epsilon_{ij}] = 0
\]

in \( D \) and therefore \( \Delta \epsilon_{ij}(x, t) \) and hence \( \Delta \nu_i(x, t) \) and \( \Delta \sigma_{ij}(x, t) \) must vanish identically in the time interval \([0, T]\) everywhere in \( D \). The last conclusion implies that there cannot exist two distinct stress and strain fields satisfying (2), (3), (4) and (5) and (15).

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**Bibliography**


**DISTORTION OF BOUSSINESQ FIELD BY CIRCULAR HOLE**

by R. M. EVAN-IWANOWSKI (Syracuse University)

**Introduction.** The classical Boussinesq solution to the problem of a concentrated load acting on the straight boundary of a semi-infinite plate is basic to a number of problems in the plane theory of elasticity. Barjansky [1] modified the Boussinesq problem and analyzed the effects of a circular hole in the plate. In the following paper the latter problem has been restated and some corrections affecting the results have been made.**

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**Calculations are shown in Appendix.*
Statement of the problem and general procedure. Consider a semi-infinite thin plate containing a circular hole and loaded by a concentrated normal force $P$ (case $2V$), Fig. 1a, or a concentrated tangential force $P$ (case $T$), Fig. 1b. The subscript 1 will refer to the case $2V$ and subscript 2 to the case $T$.

To the Boussinesq stress function, $\phi$, the stress function $\chi$ is added such that stress function $\Phi = \phi + \chi$ will satisfy the boundary conditions around the circular hole, i.e., no normal and shearing stresses there. Besides, the function $\chi$ must be chosen so that it renders no stresses at infinity and results in zero normal and shearing stresses on the straight boundary. The known Boussinesq solutions are (see Fig. 1a and 1b)

$$\phi_1 = -\frac{(P/\pi)}{2}(y - y_0) \tan^{-1} [(y - y_0)/x],$$

$$\phi_2 = -\frac{(P/\pi)}{2}x \tan^{-1} [(y - y_0)/x].$$

The problem is solved in bipolar coordinates:

$$x = -J \sinh \xi; \quad y = J \sin \eta; \quad J = a(\cosh \xi - \cos \eta)^{-1}.$$  
(2)

For the case considered here $\xi \leq 0$. The circular boundary is specified by setting $\xi = \xi_0 = \text{const}$, and the straight boundary $\xi = 0$.

Denote:

$$\beta = \tan^{-1} (y_0/a); \quad p = \sin \beta \cosh \xi; \quad q = \cos \beta \sinh \xi; \quad \psi = \eta + \beta.$$  
(3)

Thus (1) can be written in bipolar coordinates as

$$\phi_2/J = (P/\pi) \sinh \xi \tan^{-1} [(p - \sin \psi)/q]; \quad \phi_1/J = (\phi_2/J)[(p - \sin \psi)/q].$$  
(4)

General form of the stress function $\chi$ in bipolar coordinates is known, see [2], and the stress function $\chi$ satisfying the above imposed conditions is

$$\chi/J = B \xi \cosh \xi - [B(\xi - \sinh \xi \cosh \xi) + 2F \sinh^2 \xi] \cos \eta + (G' \cosh 2\xi + F') \sin \eta$$

$$+ \sum_{k=2}^{\infty} \left\{ \frac{[E_a a_k(\xi) + F_a \sigma_k(\xi)] \cos k\eta + [E'_a a_k(\xi) + F'_a \sigma_k(\xi)] \sin k\eta}{(k - 1)} \right\},$$

where $a_k(\xi) = (k - 1) \sinh \xi \sinh k\xi; \quad \sigma_k(\xi) = k \sinh \xi \cos k\xi - \cosh \xi \sinh k\xi.$

The unknown coefficients in (5) are determined from the boundary conditions around the circular hole. To this end the Boussinesq stress functions are expanded into Fourier series in $\eta$.

Fourier representation of functions $\phi$. Starting with $\phi_2$, we have

$$\phi_2/J = R_0/2 + \sum_{k=1}^{\infty} (R_k \cos k\eta + S_k \sin k\eta),$$

where

$$R_0/2 = \frac{(P \sinh \xi/\pi)}{2} \tan^{-1} (\tan \beta \cosh \xi)$$

$$+ \sinh \xi \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2\xi})],$$

$$R_k = -(P \sinh \xi/\pi)e^k(-1)^k \sin k\beta/k,$$

$$S_k = -(P \sinh \xi/\pi)[1 - (-1)^k \cos k\beta]/k.$$  
(6)

see [1], p. 21)
Fig. 1a

Fig. 1b
The Fourier coefficients of $\phi_{1}/J$ are found from (6) by multiplying $\phi_{2}/J$ by $(p - \sin \psi)/q$, see (3).

Case T. Tangential concentrated force $P$ acting in the straight boundary, Fig. 1b. The required stress function $\chi_{3}$ has the form given in (5). Thus

$$\frac{\phi_{2}}{J} = \phi_{2}/J + \chi_{3}/J,$$

the unknown coefficients in $\chi_{3}$ are found from the boundary conditions around the circular hole $\xi_{0}$. For this case:

$$F = \left(\frac{P}{\pi}\right)\left\{1 + 4 \sinh \xi_{0} \sin 2\beta/d_{1} + e^{2i\xi_{0}}/\sinh \xi_{0} - \cosh \xi_{0} \tan^{-1} (\sin 2\beta/d_{1}^{*})\right\}$$

$$B = \left(\frac{P}{2\pi \sinh^{2} \xi_{0}}\right)\left\{\sinh 2\xi_{0}(1 + 4 \sinh \xi_{0}) \sin 2\beta/d_{1}
- \cosh \xi_{0} \sinh 2\xi_{0} \tan^{-1} (\sin 2\beta/d_{1}^{*}) + e^{2i\xi_{0}}(2 \cosh \xi_{0} - 1) \sin \beta\right\}$$

$$G' = -\left(\frac{P}{\pi}\right)(1 + \cos \beta)/(1 - 4e^{-4\xi_{0}})$$

$$F' = 0$$

where

$$4(\cosh 2\xi_{0} - \cos 2\beta) = d_{1} ; \quad \cos 2\beta - e^{-2i\xi_{0}} = d_{1}^{*}.$$  (a)

The unknown coefficients for $k \geq 2$ are:

$$E_{k} = [-P(-1)^{k} \sin k\beta(k^{2} \sinh^{2} \xi_{0} + k \sinh \xi_{0} \cosh \xi_{0} - e^{-k\xi_{0}} \sinh k\xi_{0})]/k D_{k}^{*}$$

$$F_{k} = -P(-1)^{k}(k - 1) \sin k\beta \sinh^{2} \xi_{0}/D_{k}^{*}$$

$$E'_{k} = P[1 - (-1)^{k} \cos k\beta](k^{2} \sinh^{2} \xi_{0} + k \sinh \xi_{0} \cosh \xi_{0} - e^{-k\xi_{0}} \sinh k\xi_{0})/k D_{k}^{*}$$

$$F'_{k} = P(k - 1)[1 - (-1)^{k} \cos k\beta] \sinh^{2} \xi_{0}/D_{k}^{*}$$

$$D_{k}^{*} = 2\pi(\sinh^{2} k\xi_{0} + k^{2} \sinh^{2} \xi_{0}).$$

(see [1], p. 25)

Case N. Normal concentrated force $P$ acting on the straight boundary, Fig. 1a. In this case the Boussinesq stress function $\phi_{1}$ is represented as

$$\frac{\phi_{1}}{J} = T_{0}/2 + \sum_{k=1}^{\infty} (T_{k} \cos k\eta + U_{k} \sin k\eta),$$

where

$$T_{0}/2 = \left(\frac{P}{\pi}\right)\{\tan \beta \cosh \xi[\tan^{-1} (\tan \beta \csc \xi) + \sinh \xi \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2i\xi})] - e^{i}(1 + \cos \beta)/2 \cos \beta\},$$

$$T_{1} = \left(\frac{P}{\pi}\right) \tan \beta\{e^{i} \sin \beta \cosh \xi - \tan^{-1} (\tan \beta \csc \xi)
- \sinh \xi \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2i\xi})]\},$$

$$U_{1} = \left(\frac{P}{\pi}\right)\left\{\frac{1}{2} \tan \beta[1 + (1 + e^{2i}) \cos \beta] - \tan^{-1} (\tan \beta \csc \xi)
- \sinh \xi \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2i\xi})]\right\},$$

$$T_{k} = -(Pe^{ik}/\pi)\{A_{k}(\xi) + B_{k}(\xi)[1 + (-1)^{k}(\cos k\beta)/\cos \beta]\},$$

$$U_{k} = -(Pe^{ik}/\pi)\{A_{k}(\xi) - B_{k}(\xi)[(\tan \beta)/k + (-1)^{k}(\sin k\beta)/\cos \beta]\},$$

$$\text{where}$$

$$4(\cosh \xi_{0} - \cos \beta) = d_{1} ; \quad \cos 2\beta - e^{-2i\xi_{0}} = d_{1}^{*}.$$  (a)
where
\[ A_\iota(\xi) = (-1)^\iota \tan \beta \sin k\beta \cosh \xi/k, \]
\[ B_\iota(\xi) = (k \sinh \xi - \cosh \xi)/(k^2 - 1). \]

Moreover
\[ F = (P/2\pi \sinh^2 \xi_0) \{e^{k\xi} \tan \beta \sin \beta \cosh \xi_0 - (1 + \cos \beta)/2 \cos \beta - \tan \beta \sinh \xi_0 \cosh \xi_0 \]
\[ \cdot \left[ \sin 2\beta(1 + 4 \sinh \xi_0)/4(\cos 2\beta - \cosh 2\xi_0) + \cosh \xi_0 \tan^{-1}(\sin 2\beta/d^*) \right], \]
\[ B = (1/2 \sinh^2 \xi_0) \{4F \cosh \xi_0 \sinh \xi_0 + (P/\pi) \tan \beta \tan^{-1}(\sin 2\beta/d^*) \]
\[ - \sin 2\beta(1 - 4 \sinh \xi_0)/d_1 - e^{-2t_\iota \sin \beta} \}, \]
\[ G' = (P/2\pi \sinh^2 \xi_0) \{e^{2t_\iota} \sin \beta - \sin 2\beta(1 + 4 \sinh \xi_0)/d_1 \]
\[ + \cosh \xi_0 \tan^{-1}(\sin 2\beta/d^*) \}, \]
\[ F' = 0, \]
where \(d_1\) and \(d^*\) are given by (a).

The unknown coefficients \(E_k, F_k, E'_k\) and \(F'_k\) for \(k \geq 2\) are:
\[ E_k = -(P/2\pi) \{c_k \sin k\beta + k d_k \sinh \xi_0\}/D_k, \]
\[ F_k = -(P/2\pi) \{e_k \sin k\beta + d_k f_k(1 + k)^{-1}\}/D_k, \]
\[ E'_k = (P/2\pi) \{c_k \cos k\beta - k g_k \sinh^2 \xi_0\}/D_k, \]
\[ F'_k = (P/2\pi) \{e_k \cos k\beta - (k + 1)^{-1} g_k f_k\}/D_k, \]
where
\[ c_k = (-1)^k \tan \beta(k \sinh k\xi_0 \cosh \xi_0 + \sinh^2 \xi_0 - e^{k\xi_0} \sinh k\xi_0), \]
\[ d_k = [1 - (-1)^k \cos k\beta/\cos \beta], \]
\[ e_k = (-1)^k(k - 1)k^{-1} \tan \beta(k \sinh \xi_0 \cosh \xi_0 + e^{k\xi_0} \sinh k\xi_0), \]
\[ f_k = (k^2 \sinh^2 \xi_0 - k \sinh \xi_0 - e^{k\xi_0} \sinh k\xi_0), \]
\[ g_k = [k^{-1} \tan \beta + (-1)^k \sin k\beta/\cos \beta], \]
\[ D_k = (\sinh^2 k\xi_0 - k\sinh^2 \xi_0). \]

Appendix

Case \(k = 0\). Calculation of \(R_0\), for \(\beta \neq 0\), (see [1], p. 28)
\[ R_0 = (P \sinh \xi/\pi^2) \int_{-\pi}^\pi \tan^{-1} [(p - \sin \psi)/q] e^{ik\psi} = (p/\pi^2) \sinh \xi \cdot J_0. \]

Integrating \(J_0\) by parts we get
\[ I_0 = [\psi \tan^{-1} ((p - \sin \psi)/q)]_{-\pi}^\pi + \int_{-\pi}^\pi \psi \cos \psi \, d\psi/(p - \sin \psi)^2 = 2\pi \tan^{-1} (p/q) \cdot J_0. \]
\( J_0 \) can be represented as
\[
J_0 = -(2 \cos 2\beta)^{-1} \int_{-\pi}^{\pi} \psi[(e^{i\psi} - e^{i+\beta})^{-1} - (e^{i\psi} - e^{-i+\beta})^{-1} \\
- (e^{i\psi} + e^{-i+\beta})^{-1} + (e^{i\psi} + e^{-i-\beta})^{-1}] e^{i\psi} \, d\psi.
\]

A typical integral is:
\[
\int_{-\pi}^{\pi} \psi e^{i\psi}(e^{i\psi} - c_0) \, d\psi.
\]

To evaluate these integrals, a complex variable \( \xi \) is introduced
\[
\xi = \psi + i\omega
\]

and the integration proceeds along the contour, \( C \), shown in the figure. Single poles \( \xi_1 = \beta - i\xi \) and \( \xi_2 = \pi - \beta - i\xi \) of the first and second terms are located within the contour of integration as shown in the figure, and the second and fourth term have single poles outside of the contour of integration. Thus
\[
\int_C = \int_{V_1} + \int_{V_2} + \int_{\omega=X} + \int_{-\pi}^{\pi}
\]
is equal to \( 2\pi i \text{ Res } (\xi = \beta - i\xi) \) for the first integral; is equal to \( 2\pi i \text{ Res } (\xi = \pi - \beta - i\xi) \) for the third integral and zero for the second and fourth. Here \( V_1 \) and \( V_2 \) are the vertical segments of the integration contour. On \( V_1 \), \( \xi = \pi + i\omega \) and on \( V_2 \), \( \xi = -\pi + i\omega \).

It is a simple matter to demonstrate that the integrals for \( \omega = \Lambda \to \infty \) tend to zero. Obviously the last integrals represent parts of \( J_0 \). Now we proceed to evaluate the integrals along \( V_1 \) and \( V_2 \).

\[
\int_{V_1} - \int_{V_2} = \int_0^\Lambda (\pi + i\omega)e^{i(\pi + i\omega)}[e^{i(\pi + i\omega)} - c_0]\^{-1} \, d(\omega) \\
- \int_0^\Lambda (-\pi + i\omega)e^{i(-\pi + i\omega)}[e^{i(-\pi + i\omega)} - c_0]\^{-1} \, d(\omega)
\]

\[
= 2\pi i \int_0^\Lambda e^{-\omega}(e^{-\omega} + c_0)^{-1} \, d\omega = 2\pi i[\ln (e^{-\omega} + c_0)]_0^\Lambda.
\]
Letting $\Lambda \to \infty$, we get
\[
\lim_{\Lambda \to \infty} 2\pi i \ln (e^{-\alpha} + e^\alpha) = 2\pi i [\ln c_k - \ln (1 + c_k)] = 2\pi i \ln [c_k/(1 + c_k)].
\]

Summing up for all $k (k = 1, 2, 3, 4)$ we get
\[
\sum_{k=1}^{4} \int_{c} = 2 \cos \beta \cdot J_0 + 2\pi i [2i\beta + 2i \tan^{-1} \sin 2\beta/(\cos 2\beta - e^{-2\beta})] = 2\pi i (2i\beta),
\]
where $\text{Res} (f_1) + \text{Res} (f_2) = 2i\beta$. Thus
\[
J_0 = (\pi/\cos \beta) \tan^{-1} \sin 2\beta/(\cos 2\beta - e^{-2\beta})
\]
and consequently
\[
I_0 = 2\pi \{\tan^{-1} (\tan \beta \coth \xi) + \sinh \xi \tan^{-1} \sin 2\beta/(\cos 2\beta - e^{-2\beta})\}
\]
and finally
\[
R_0/2 = (P \sinh \xi/\pi) \{\tan^{-1} (\tan \beta \coth \xi) + \sinh \xi \tan^{-1} \sin 2\beta/(\cos 2\beta - e^{-2\beta})\}.
\]

References