

FINITE PURE BENDING OF CIRCULAR CYLINDRICAL TUBES*

By

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1. Introduction. The present paper is concerned with the determination of stresses, deformations and stiffness of originally straight circular tubes in pure bending. The non-linear problem of determining the stiffness of such a tube as a function of the applied moment and the determination of a critical moment for which flattening instability occurs has originally been discussed by Brazier [1].

An alternate more precise formulation of the problem of flattening instability of circular cross-section tubes is contained in a recent paper by one of the present authors [2], as a special case of results for pure bending of general cylindrical tubes. In this same paper approximate solutions of the non-linear differential equations of the problem were obtained as expansions in powers of a dimensionless parameter α . It was found that the first terms of these expansions give the results of linear theory and that consideration of two terms gave the results of Brazier [1]. It was further found that consideration of three terms lead to results which differed from Brazier's to the order of ten per cent. Since the calculation of additional terms in the α -series becomes progressively more complicated, an alternate determination of the results is of interest. The present paper presents such an alternate determination, involving the iterative solution of a system of two simultaneous non-linear integral equations. In addition to this, the previous three-term α -series are extended by the calculation of fourth terms. Our calculations lead to the note-worthy conclusion that Brazier's results for flattening instability are quite close to the results of precise calculations based on the equations given in [2], in the sense that consideration of three and even four terms in the α -series lead to results which are further from the correct results (in the critical α -range) than the results based on only two terms in the α -series.

In addition to these conclusions for the problem of the flattening instability, we obtain in what follows quantitative results for the non-linear behavior of stresses and deformations in the tube. We find, in particular, that when the applied bending moment is of the order of the critical moment, the order of magnitude of the secondary circumferential wall bending stresses—associated with the flattening of the cross-section—is the same as the order of magnitude of the primary longitudinal direct fiber stresses in the tube.

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2. Basic equations. It has been shown previously [2] that the problem of pure bending of a tube with cross-section before deformation given by $x = b \sin \xi, y = -b \cos \xi$ for $0 \leq \xi \leq 2\pi$ is associated with two simultaneous non-linear differential equations for a stress function variable F and an angular displacement variable β of the following form

$$\frac{D}{b^2} \frac{d^2 \beta}{d\xi^2} = \frac{F \sin(\xi + \beta)}{R}, \quad \frac{A}{b^2} \frac{d^2 F}{d\xi^2} = \frac{\cos(\xi + \beta)}{R} \tag{1a, b}$$

In these equations R is the radius of curvature of the originally straight axis of the tube, $D = E_b h^3 / 12$ is the circumferential wall bending stiffness factor and $1/A = E_b h$ is the axial stretching stiffness factor of the tube.*

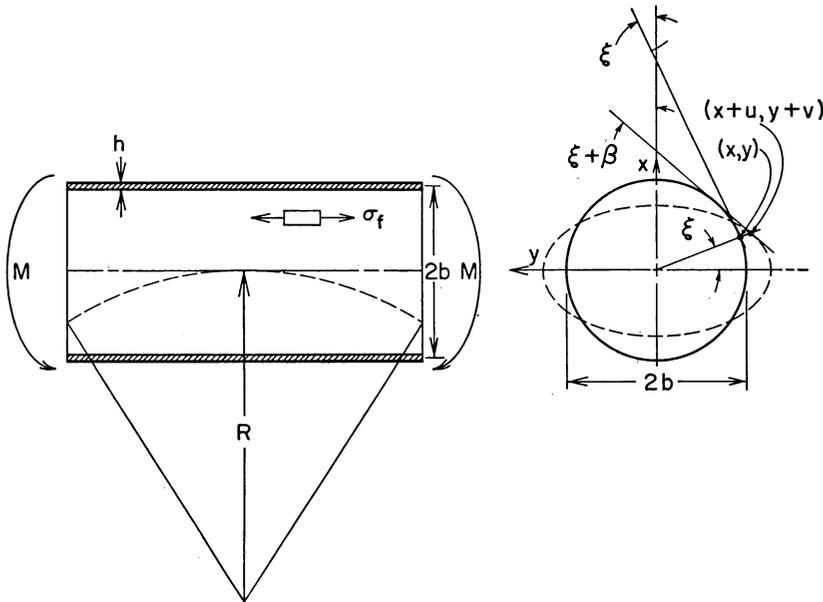


FIG. 1. Notation

Equations (1) are to be solved in the interval $0 \leq \xi \leq \frac{1}{2}\pi$ subject to the following boundary conditions,

$$\frac{dF(0)}{d\xi} = F\left(\frac{\pi}{2}\right) = \beta(0) = \beta\left(\frac{\pi}{2}\right) = 0 \tag{2}$$

To be determined are in particular the applied moment M given by

$$M = -4b \int_0^{\pi/2} F \cos(\xi + \beta) d\xi \tag{3}$$

*In our earlier paper [2] it had been assumed that $D = E_b h^3 / 12(1 - \nu^2)$ which is the appropriate circumferential stiffness factor for small cylindrical bending. Considering that in the range of practical interest we will have cylindrical bending with relatively large deflections a stiffness factor D without the term $(1 - \nu^2)$ seems more appropriate.

axial fiber stress σ_f and circumferential bending stress σ_b given by

$$\sigma_f = \frac{1}{hb} \frac{dF}{d\xi}, \quad \sigma_b = \frac{E_b h}{2b} \frac{d\beta}{d\xi} \quad (4a, b)$$

and cross sectional flattening and bulging displacements given by

$$\frac{u}{b} = \int_0^{\pi/2} \cos(\xi + \beta) d\xi - 1, \quad \frac{v}{b} = 1 - \int_0^{\pi/2} \sin(\xi + \beta) d\xi \quad (5a, b)$$

Equations (1) to (3) are made non-dimensional by setting

$$F = \frac{b^2}{AR} f, \quad \alpha = \frac{b^2}{\sqrt{AD}} \frac{1}{R}, \quad m = \frac{M}{\pi b} \sqrt{\frac{A}{D}} \quad (6a, b, c)$$

Indicating differentiation with respect to ξ by primes, we now have the differential equations

$$\beta'' = \alpha^2 f \sin(\xi + \beta), \quad f'' = \cos(\xi + \beta) \quad (7a, b)$$

with boundary conditions

$$f'(0) = f\left(\frac{\pi}{2}\right) = \beta(0) = \beta\left(\frac{\pi}{2}\right) = 0 \quad (8)$$

The dimensionless moment m becomes, after an integration by parts

$$m = \frac{4\alpha}{\pi} \int_0^{\pi/2} (f')^2 d\xi, \quad (9)$$

and dimensionless stresses may be obtained in the form

$$\frac{\sigma_f b}{E_s h} = \alpha f', \quad \frac{\sigma_b b}{E_b h} = \frac{1}{2} \beta' \quad (9a, b)$$

Two alternative dimensionless stress quantities may be defined as follows. One definition makes use of the maximum fiber stress $\sigma_f^{(0)} = E_s b/R$ which would exist in the tube bent to a radius R if there were no flattening effect. Introduction of $\sigma_f^{(0)}$ into Eqs. (4) and (6) leads to the formulas

$$\frac{\sigma_f}{\sigma_f^{(0)}} = \frac{df}{d\xi}, \quad \frac{\sigma_b}{\sigma_f^{(0)}} = \frac{E_b}{E_s} \frac{\sqrt{3}}{\alpha} \frac{d\beta}{d\xi} \quad (10a, b)$$

As flattening makes the tube more flexible than it would otherwise be, we expect that the ratio of $\sigma_f/\sigma_f^{(0)}$ will decrease as α increases.

A second non-dimensionalization makes use of the maximum fiber stress $\sigma_f^{(1)} = Mb/I$, where $I = \pi b^3 h$, which would exist in the tube subject to an applied moment M if there were no flattening effect. Introduction of $\sigma_f^{(1)}$ into Eqs. (4) and (6) leads to the formulas

$$\frac{\sigma_f}{\sigma_f^{(1)}} = \frac{\alpha}{m} \frac{df}{d\xi}, \quad \frac{\sigma_b}{\sigma_f^{(1)}} = \frac{E_b}{E_s} \frac{\sqrt{3}}{m} \frac{d\beta}{d\xi} \quad (11a, b)$$

In these equations we consider α a function of m which is defined by means of Eq. (9). Whether or not $\sigma_f/\sigma_f^{(1)}$ will increase or decrease with increasing m will depend on the shape of the curve for $\sigma_f/\sigma_f^{(1)}$ as function of ξ and cannot be predicted without numerical calculations. Numerical calculations are also needed for a comparison of the magnitude

of the secondary bending stresses σ_b with the magnitude of the primary direct fiber stress σ_r , in their dependence on α or m .

3. Expansion in powers of α . The boundary value problem (7) and (8) may be solved, as in [2], by expansions

$$f = f_0 + \alpha^2 f_2 + \dots, \quad \beta = \alpha^2 \beta_2 + \alpha^4 \beta_4 + \dots \tag{12a, b}$$

Expanding $\sin \beta$ and $\cos \beta$ in terms of α^2 we obtain a system of successive linear differential equations, of which we list the first seven as follows:

$$f_0'' = \cos \xi, \quad \beta_2'' = f_0 \sin \xi, \quad f_2'' = -\beta_2 \sin \xi \tag{13a, b, c}$$

$$\beta_4'' = f_2 \sin \xi + f_0 \beta_2 \cos \xi, \quad f_4'' = -\beta_4 \sin \xi - \frac{1}{2} \beta_2^2 \cos \xi \tag{14a, b}$$

$$\beta_6'' = (f_4 - \frac{1}{2} f_0 \beta_2^2) \sin \xi + (f_2 \beta_2 + f_0 \beta_4) \cos \xi \tag{15}$$

$$f_6'' = -(\beta_6 - \frac{1}{6} \beta_2^3) \sin \xi - \beta_2 \beta_4 \cos \xi \tag{16}$$

We require that the boundary conditions (8) be satisfied identically in α .

The functions $f_0, f_2, f_4, \beta_2, \beta_4$ have been calculated in [2] and are listed here for completeness sake. The functions f_6 and β_6 as well as formulas for stresses and displacements have not been obtained before. We find

$$f_0 = -\cos \xi, \quad \beta_2 = \frac{1}{8} \sin 2\xi, \quad f_2 = \frac{1}{18} \cos \xi (1 + \frac{1}{2} \sin^2 \xi) \tag{17a, b, c}$$

$$\beta_4 = \frac{1}{384} \sin 2\xi \left(1 + \frac{10}{3} \cos^2 \xi \right), \tag{18a}$$

$$f_4 = \frac{439}{64800} \cos \xi \left(1 + \frac{1}{2} \sin^2 \xi - \frac{126}{439} \sin^4 \xi \right) \tag{18b}$$

$$\beta_6 = \frac{121}{69120} \sin 2\xi \left(-1 - \frac{100}{363} \cos^2 \xi + \frac{276}{605} \cos^4 \xi \right) \tag{19}$$

$$f_6 = \frac{580399}{12150 \cdot 117600} \cos \xi \left(-1 - \frac{1}{2} \sin^2 \xi - \frac{2191761}{2321596} \sin^4 \xi + \frac{249075}{580399} \sin^6 \xi \right) \tag{20}$$

Insertion of Eq. (12a) into Eq. (9) gives an expression for the moment function m of the form

$$m = \frac{4\alpha}{\pi} \int_0^{\pi/2} \{ (f_0')^2 + 2f_0' f_2' \alpha^2 + [(f_2')^2 + 2f_0' f_4'] \alpha^4 + (2f_2' f_6' + 2f_2' f_4') \alpha^6 + \dots \} d\xi \tag{21}$$

Substitution of (17a), (17c), (18b) and (20) and subsequent integration gives the relation

$$m = \alpha - \frac{1}{8} \alpha^3 - \frac{1}{96} \alpha^5 + \frac{163}{82944} \alpha^7 + \dots \tag{22}$$

The partial sum obtained by omitting the last listed term in Eq. (22) agrees with the result given in [2]. Retaining only the first two terms on the right side of Eq. (22) gives the result of Brazier. A quantitative discussion of the dimensionless moment curvature relation (22) is given further on in conjunction with a discussion of the corresponding relation obtained from the numerical solution of the integral equation.

For the calculation of stresses in accordance with Eqs. (9a, b), (10) and (11) we need the following expressions for the derivatives f' and β' :

$$f' = \sin \xi - \frac{\alpha^2}{12} \sin^3 \xi + \frac{\alpha^4}{144} \sin^3 \xi \left(\frac{7}{5} \sin^2 \xi - \frac{31}{12} \right) + \frac{\alpha^6}{4800} \sin^3 \xi \left(-\frac{47}{7} \sin^4 \xi + \frac{7681}{540} \sin^2 \xi - \frac{719}{162} \right) + \dots \quad (23)$$

$$\beta' = \frac{\alpha^2}{2} \left(\cos^2 \xi - \frac{1}{4} \right) + \frac{\alpha^4}{24} \left(\frac{5}{3} \cos^4 \xi - \cos^2 \xi - \frac{1}{8} \right) + \frac{\alpha^6}{480} \left(\frac{23}{5} \cos^6 \xi - \frac{307}{54} \cos^4 \xi - \frac{71}{36} \cos^2 \xi + \frac{121}{72} \right) + \dots \quad (24)$$

which follow from Eqs. (12) and (17) to (20).

Expressions for the flattening and bulging displacements follow from Eqs. (5), (12) and (17) to (20) in the form

$$-\frac{u}{b} = \frac{\alpha^2}{12} + \frac{71\alpha^4}{8640} + \frac{44551\alpha^6}{7560 \cdot 7200} + \dots \quad (25)$$

$$-\frac{v}{b} = \frac{\alpha^2}{12} + \frac{\alpha^4}{960} - \frac{2059\alpha^6}{168 \cdot 7200} + \dots \quad (26)$$

We note that while for sufficiently small α we have $u \doteq v$, it is found that for increasing α the flattening displacement u increases more rapidly than the bulging displacement v .

4. Integral equation formulation and numerical solution. The differential equations (7) with boundary conditions (8) may be reduced to the following system of integral equations

$$\bar{f}(x) = -\left(\frac{1}{2}\pi\right)^2 \int_0^1 \cos \left[\frac{1}{2}\pi y + \bar{\beta}(y) \right] G_f(x, y) dy \quad (27)$$

$$\bar{\beta}(x) = -\left(\frac{1}{2}\pi\alpha\right)^2 \int_0^1 \bar{f}(y) \sin \left[\frac{1}{2}\pi y + \bar{\beta}(y) \right] G_\beta(x, y) dy \quad (28)$$

where $x = \xi/\frac{1}{2}\pi$, $\bar{f}(x) = f(\frac{1}{2}\pi x)$, $\bar{\beta}(x) = \beta(\frac{1}{2}\pi x)$, and G_f , G_β are Green's functions given by

$$G_f(x, y) = \begin{cases} 1 - y & x \leq y \\ 1 - x & x \geq y \end{cases} \quad (29)$$

$$G_\beta(x, y) = \begin{cases} (1 - y)x & x \leq y \\ y(1 - x) & x \geq y \end{cases} \quad (30)$$

(in what follows the bars on \bar{f} , $\bar{\beta}$ will not be written, for simplicity's sake.) Numerical solutions of adequate accuracy of these integral equations may be obtained by a combination of iteration and numerical integration.

Values of the dimensionless moment m and of the displacements u and v are calculated by introducing the solutions $f(x)$, $\beta(x)$ into the integrals in Eqs. (3) and (5). Dimensionless stresses in accordance with Eqs. (9a, b), (10) and (11) are obtained by calculating $d\beta/dx$ and df/dx in terms of the integrals which follow, rather than by numerical differen-

tiation of the discrete values of f and β obtained from the solution of Eqs. (27) and (28). We find, in dimensionless form

$$m = -2\alpha \int_0^1 \cos [\frac{1}{2}\pi y + \beta(y)]f(y) dy \tag{31}$$

$$\frac{df}{dx} = \frac{1}{4} \pi^2 J_3(x) \tag{32}$$

$$\frac{d\beta}{dx} = \frac{1}{8} (\pi\alpha)^2 [J_1(x) - J_0] \tag{33}$$

$$\frac{u}{b} = \frac{\pi}{2} J_3(x) - \sin \frac{\pi}{2} x, \quad \frac{v}{b} = -\frac{\pi}{2} J_2(x) + \cos \frac{\pi}{2} x \tag{34a, b}$$

where

$$J_0 = \int_0^1 yf(y) \sin [\frac{1}{2}\pi y + \beta(y)] dy \tag{35}$$

$$J_1(x) = \int_x^1 f(y) \sin [\frac{1}{2}\pi y + \beta(y)] dy \tag{36}$$

$$J_2(x) = \int_x^1 \sin [\frac{1}{2}\pi y + \beta(y)] dy \tag{37}$$

$$J_3(x) = \int_0^x \cos [\frac{1}{2}\pi y + \beta(y)] dy \tag{38}$$

In order to describe the iteration scheme to be used, we write Eqs. (27) and (28) in the form

$$f = K_1[\beta], \quad \beta = \alpha^2 K_2[f, \beta] \tag{39a, b}$$

where the integral operator K_1 depends on β only and where K_2 is linear in f . The most straightforward iteration of (39) is expressed by $f_{n+1} = K_1[\beta_n]$, $\beta_{n+1} = \alpha^2 K_2[f_n, \beta_n]$. Using the iteration f_{n+1} for the calculation of β_{n+1} , a more rapidly converging iteration scheme for (39) is

$$f_{n+1} = K_1[\beta_n], \quad \beta_{n+1} = \alpha^2 K_2[f_{n+1}, \beta_n] \tag{40a, b}$$

The two equations (40) can be written as one equation as follows

$$\beta_{n+1} = \alpha^2 K_2\{K_1[\beta_n], \beta_n\} \equiv \alpha^2 T[\beta_n] \tag{41}$$

A proof of the convergence of this iteration (which implies existence and uniqueness of the solution of Eqs. (39)) in the range $0 \leq \alpha^2 \leq 1.8$ is given in the Appendix. Numerical calculation shows convergence of (41) for values of α^2 up to about 5. For larger α^2 , examples of both oscillations and steady increase in magnitude of successive iterates were obtained, that is, the iteration scheme diverges.

In order to obtain solutions for larger values of α^2 , the iteration scheme is modified by introduction of two "relaxation parameters" λ and μ as follows¹

$$\begin{aligned} f_{n+1} &= K_1[\beta_n], & f_{n+1}^* &= \lambda f_{n+1} + (1 - \lambda)f_n \\ \beta_{n+1}^* &= \alpha^2 K_2[f_{n+1}^*, \beta_n], & \beta_{n+1} &= \mu \beta_{n+1}^* + (1 - \mu)\beta_n \end{aligned} \tag{42}$$

¹A similar iteration scheme using one relaxation parameter was employed by Keller and Reiss in solving a system of non-linear difference equations [3].

Clearly, if the sequences f_n, β_n converge, they converge to a solution of (39). The relaxation parameters λ and μ are allowed to depend on α^2 . With this scheme and with appropriate choice of λ, μ , the speed of convergence was considerably increased as compared with the iteration (40), and convergence was induced for values of α^2 for which (40) diverges. In this way solutions up to $\alpha^2 \doteq 20$ were obtained; the range could probably be extended to still larger α^2 by proper choice of λ, μ , although solutions beyond a critical value α_c^2 (see Section 5) are physically less interesting. The following table shows some numerical results. The special case $\lambda = \mu = 1$ is identical with the iteration (40). A proof for the convergence of the modified scheme (42) for values of $\alpha^2 > 1.8$ has yet to be obtained.

Table 1. Number of iterations for 0.1% accuracy of solutions of (39)

α^2	$\mu = \lambda = 1$	$\lambda = 1, \mu = \frac{3}{4}$	$\lambda = \frac{3}{4}, \mu = 1$	$\lambda = \frac{3}{4}, \mu = \frac{3}{4}$	$\lambda = .65, \mu = .60$
2.6	6	8	9		
6.0	34	10	18		
9.0	oscill.	32	20	16	
13.0	div.	div.	—	32	27
20.0	div.	div.	oscill.	—	33

In the execution of the iteration schemes (40) and (42) the five point Newton-Côtes formula

$$\int_{x_0}^{x_5} y(x) dx = \frac{5}{288} \Delta x (19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5) + E_5$$

$$|E_5| \leq \frac{275}{12096} (\Delta x)^7 \text{Max}_{x_0 \leq x \leq x_5} |y^{(v)}(x)| \quad (43)$$

was used for the numerical integration. In the Appendix it is shown how to obtain explicit bounds on error terms such as E_5 by means of certain inequalities satisfied by the solution of (39) (usually referred to as "a priori estimates"). It turned out that step sizes between $\Delta x = 1/20$ and $\Delta x = 1/50$ were needed to obtain an accuracy of 3 to 4 digits in the numerical solutions for f and β .¹

Some numerical solutions of Eqs. (39) have previously been calculated by G. L. Brown [4], who used Simpson's rule for numerical integration combined with an iteration equivalent to (40). He found, using interval lengths $\Delta x = \frac{1}{5}, \frac{1}{8}$ and $\frac{1}{18}$ that the latter was not small enough to draw conclusions on the accuracy of the results obtained.

5. Discussion of results. The integral equations (39) have been solved for values of α^2 up to 25. These solutions are in the following referred to as "numerical solutions." A comparison of the α -expansion with the numerical solutions shows that the α -expansion solutions are accurate almost up to the critical value α_c , defined by $dm/d\alpha = 0$ (see Figs. 2-4). For larger values of α , they become quite inaccurate. As a check on the

¹The calculations were carried out on the IBM-709s at Western Data Processing Center, Univ. of Calif., Los Angeles, and at Computation Center, M.I.T., Cambridge.

numerical solutions, Eqs. (7) were approximated by a finite difference system

$$\begin{aligned} \beta_{n+1} + \beta_{n-1} - 2\beta_n &= h^2 \alpha^2 \sin(\xi_n + \beta_n) f_n & n = 0, 1, \dots, N. \\ f_{n+1} + f_{n-1} - 2f_n &= h^2 \cos(\xi_n + \beta_n) \end{aligned} \quad (44)$$

which was solved for $\beta_i, f_i, i = 2, \dots, N$, the values $f_0, f_1, \beta_0, \beta_1$ being given from the

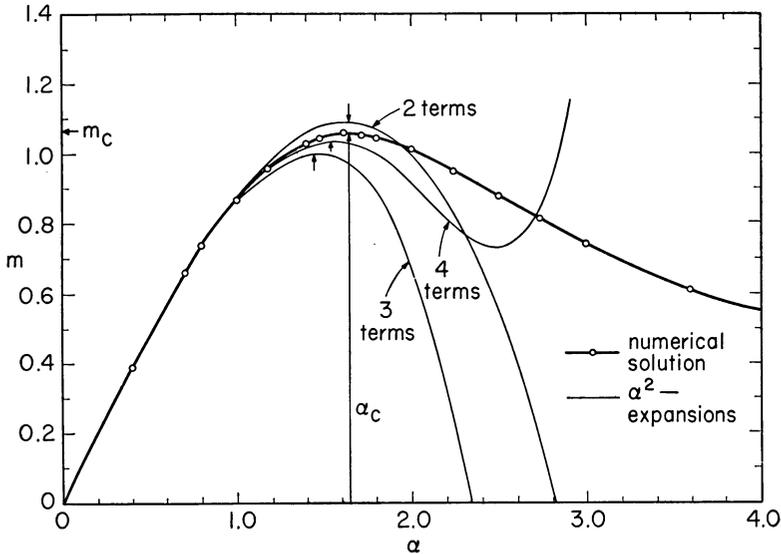


FIG. 2. Dimensionless Moment Curvature Relation

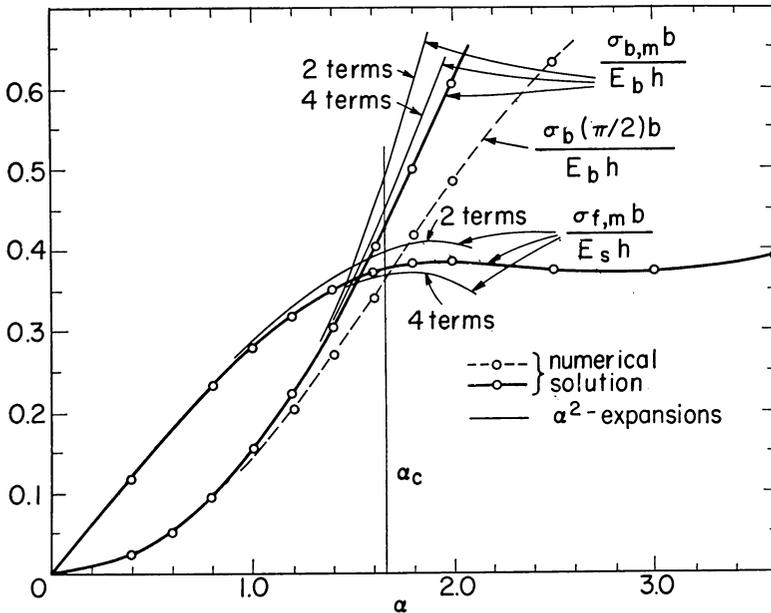


FIG. 3. Dimensionless Maximum Direct and Bending Stresses

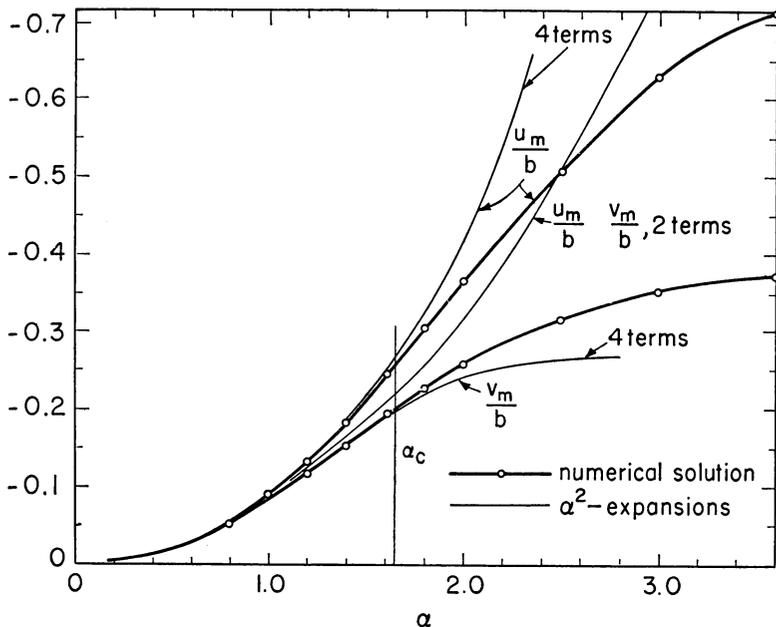


FIG. 4. Dimensionless Maximum Flattening and Bulging Displacements

solution of the integral equations. If the solutions of Eqs. (44) differed by more than one unit in the third figure from the solutions of Eqs. (39), the latter solutions were recalculated with a smaller spacing h , and the check via Eqs. (44) was repeated for that spacing.

The moment curvature relation is shown in Fig. 2. What is of particular interest is the value α_c , for which flattening instability occurs. For this value of the dimensionless curvature, the moment m attains its maximum value m_c . The numerical values for α_c and m_c when retaining 2, 3, or 4 terms in Eqs. (22), and the corresponding values α_c, m_c from the numerical solution are given below. The numbers of the last column of Table 2

Table 2. Critical curvatures and moments

	2 terms	3 terms	4 terms	numerical solution
α_c	1.513	1.422	1.468	1.526
m_c	1.086	0.998	1.034	1.063

were obtained by interpolation from a large scale plot of the dimensionless moment-curvature relation near the critical point (α_c, m_c) .

Figure 3 shows the maximum values of the dimensionless direct stress $\sigma_b/E_b h$ and bending stress $\sigma_b/E_b h$ as defined by Eqs. (9a, b). The maximum bending stress occurs at the neutral plane, that is $\sigma_{b,m} = \sigma_b(0)$. The bending stress $\sigma_b(\frac{1}{2}\pi)$ at the farthest

distance from the neutral plane is slightly less in absolute value than $\sigma_b(0)$ and is also displayed in Fig. 3.

For small α bending stresses are negligible with respect to direct fiber stresses. For values of α approaching α_c , the bending stresses become of the same order of magnitude as the direct stresses. It is interesting to note that the maximum direct stress is not attained at $\alpha_c \doteq 1.66$ but for the somewhat larger value $\alpha \doteq 2.0$.

Values of the dimensionless maximum flattening and bulging displacements of the cross section are displayed in Fig. 4. For small values of α , the two displacements are nearly identical. As in the α -expansions (Eqs. (25) and (26)), flattening increases faster with increasing α than bulging, the rate of increase being strongest near α_c for both flattening and bulging.

Next we compare our results with those of elementary linear beam theory. In Fig. 5

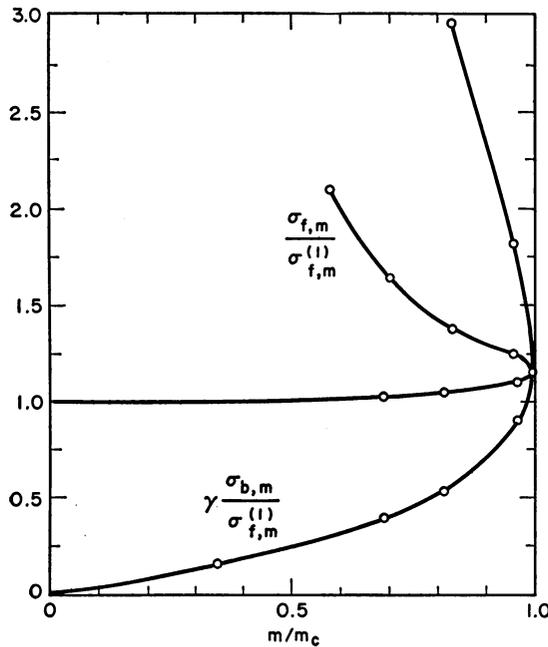


FIG. 5. Stress Ratios as Functions of Dimensionless Moment m/m_c .

the ratios $\sigma_{f,m}/\sigma_{f,m}^{(1)}$ and $\gamma\sigma_{b,m}/\sigma_{f,m}^{(1)}$ with $\gamma = E_s/E_b$, according to Eqs. (11)¹ are plotted against the dimensionless moment m/m_c . In Fig. 6, the stress ratios $\sigma_{f,m}/\sigma_{f,m}^{(0)}$ and $\gamma\sigma_{b,m}/\sigma_{f,m}^{(0)}$ are plotted against α . We conclude from these graphs that for a given moment, the direct stress produced according to the nonlinear theory is slightly larger than the value given by the linear theory which neglects the flattening of the cross section. On the other hand, for a given curvature of the central axis of the tube, the nonlinear theory stress is less than that given by the elementary beam theory.

Finally, we consider briefly the stress distribution over the cross-section of the tube and its deviation from the elementary (linear) stress distribution. As is seen from Fig. 7,

¹The subscript m refers to maximum.

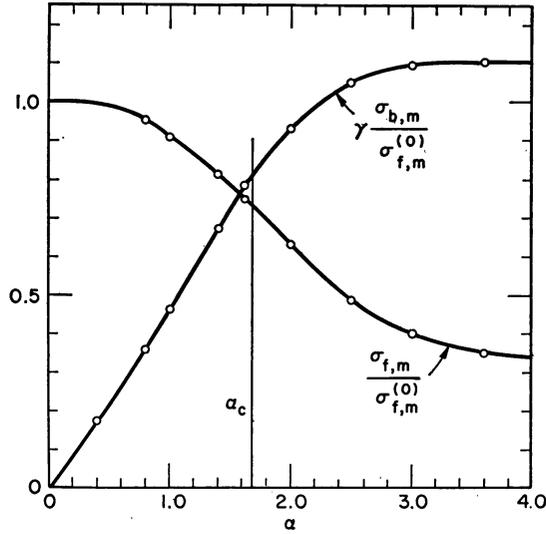


FIG. 6. Stress Ratios as Functions of Dimensionless Curvature α

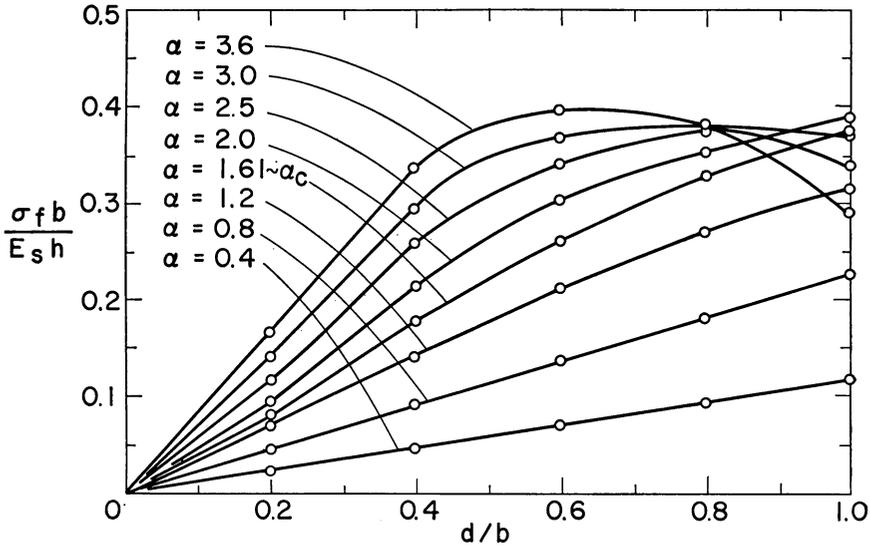


Fig. 7. Dimensionless Direct Stress vs. Dimensionless Distance d/b from Neutral Axis for Several Values of α

this deviation is quite small for values of $\alpha \leq \alpha_c$. For larger values the distribution becomes more markedly nonlinear, in fact, for $\alpha > 2.5$ the maximum fiber stress is no longer attained in the outermost fiber.

APPENDIX

In part A of the Appendix some estimates for $\beta(x)$ and $f(x)$ are derived that are used in part B for a proof of the convergence of the iteration (41) for $0 \leq \alpha^2 \leq 1.8$. In part

C the estimates are used to obtain bounds for errors introduced by replacing the exact successive integrations in the iteration (41) by approximate numerical integrations.

A. *Some Elementary Estimates for $\beta(x)$ and $f(x)$.* Replacing $|\cos(\beta(y) + y\pi/2)|$ and $|\sin(\beta(y) + \pi/2y)|$ in the integral equations (27) and (28) by unity and using the positivity of the Green's functions G_f and G_β , one obtains, for arbitrary α ,

$$|f(x)| \leq \left(\frac{\pi}{2}\right)^2 \int_0^1 G_f(x, y) dy = \left(\frac{\pi}{2}\right)^2 \left(\int_0^x (1-x) dy + \int_x^1 (1-y) dy \right) = \left(\frac{\pi}{2}\right)^2 \frac{1-x^2}{2} \tag{I}$$

and similarly, using (I)

$$|\beta(x)| \leq \left(\frac{\pi\alpha}{2}\right)^2 \int_0^1 |f(y)| G_\beta(x, y) dy \leq \left(\frac{\pi\alpha}{2}\right)^2 \frac{\pi^2}{8} \left(\int_0^x (1-y^2)(1-x)y dy + \int_x^1 (1-y^2)(1-y)x dy \right) = \frac{\pi^4 \alpha^2}{32} \left(\frac{5}{12}x - \frac{1}{2}x^2 + \frac{1}{12}x^4 \right) \leq .278\alpha^2 \tag{II}$$

For a restricted range of values of α some results on the signs of $f(x)$, $f'(x)$ and $\beta(x)$ may be obtained as follows. We have

$$\cos\left(\frac{1}{2}\pi y + \beta(y)\right) \geq 0 \tag{a}$$

as long as

$$\left| \frac{1}{2}\pi y + \beta(y) \right| \leq \frac{\pi}{2},$$

or, with inequality (II),

$$Q(y) \equiv \frac{1}{2}\pi y + \frac{1}{384} \alpha^2 \pi^4 (5y - 6y^2 + y^4) \leq \frac{\pi}{2} \tag{b}$$

Clearly, $Q(0) = 0$, $Q(1) = \frac{1}{2}\pi$, hence (a) is true if $Q'(y) \geq 0$. But $Q'(0) > 0$ and $Q'(1) \geq 0$ if $\alpha^2 \leq 384/6\pi^3$, as is easily verified. Since $Q''(y) \leq 0$ for $0 \leq y \leq 1$, we conclude $Q'(y) \geq 0$ as long as $\alpha^2 \leq 64/\pi^3 \doteq 2.05$. This proves (b) and (a). From Eq. (27) and from

$$f'(x) = \left(\frac{\pi}{2}\right)^2 \int_0^x \cos\left(\frac{\pi}{2}y + \beta(y)\right) dy$$

it follows now that

$$f(x) \leq 0, \quad f'(x) \geq 0, \quad \text{for } 0 \leq \alpha^2 \leq 2.05. \tag{III}$$

A similar argument applied to (28) yields

$$\beta(x) \geq 0, \quad \text{for } 0 \leq \alpha^2 \leq 1.25. \tag{IV}$$

The above inequalities can be combined in such a manner as to give upper and lower bounds for the moment m and the stresses without solving the integral equations.

B. *Convergence of the Iteration (41) for $0 \leq \alpha^2 \leq 1.8$.*

The principle of the convergence proof is simply to show that the transformation $\alpha^2 T$ (Eq. (41)) is contractive in the sense that

$$\alpha^2 \|T[\beta_1] - T[\beta_2]\| < k \|\beta_1 - \beta_2\|, \quad k < 1 \tag{c}$$

for all β_1, β_2 and $0 \leq \alpha^2 \leq 1.8$, by using the estimates for the solutions f, β of Eqs. (39) derived in part A. The notation adopted in (c) is

$$|| \beta(x) || = \text{Max}_{0 \leq x \leq 1} | \beta(x) | .$$

By adding and subtracting appropriate terms, we have, setting $\tilde{\beta}_i(y) = \beta_i(y) + \pi/2y$,

$$T[\beta_1] - T[\beta_2] = \frac{\pi^4}{16} \left\{ \int_0^1 (\sin \tilde{\beta}_1(y) - \sin \tilde{\beta}_2(y)) G_\beta(x, y) \int_0^1 \cos \tilde{\beta}_i(\eta) G_r(y, \eta) d\eta dy \right. \\ \left. + \int_0^1 \sin \tilde{\beta}_2(y) G_\beta(x, y) \int_0^1 G_r(y, \eta) (\cos \tilde{\beta}_1(y) - \cos \tilde{\beta}_2(y)) d\eta dy \right\} \quad (d)$$

The first term of (d) can be estimated from above by

$$\frac{\pi^4}{16} || \beta_1 - \beta_2 || \text{Max}_{0 \leq x \leq 1} \left\{ \int_0^1 G_\beta(x, y) \int_0^1 G_r(y, \eta) d\eta dy \right\} \quad (e)$$

For the estimation of the second term use is made of inequality II of part A, which gives

$$| \sin \tilde{\beta}_2(y) | < \frac{\pi}{2} y + .278\alpha^2$$

Hence, using again the Lipschitz property of the cosine-function, the following upper bound for the second term is obtained:

$$\frac{\pi^4}{16} || \beta_1 - \beta_2 || \text{Max}_{0 \leq x \leq 1} \left\{ \int_0^1 G_\beta(x, y) \left(\frac{\pi}{2} y + .278\alpha^2 \right) \int_0^1 G_r(y, \eta) d\eta dy \right\} \quad (f)$$

Substitution of (29) and (30) into the double integrals appearing in (e) and (f), and integrating, one finds

$$\int_0^1 G_\beta(x, y) \int_0^1 G_r(y, \eta) d\eta dy = \frac{1}{24} x(1-x)(5-x-x^2) \\ \frac{\pi}{2} \int_0^1 y G_\beta(x, y) \int_0^1 G_r(y, \eta) d\eta dy = \frac{\pi}{240} x(1-x^2)(7-3x^2) \quad (g)$$

Finding the maxima of these polynomials for $0 \leq x \leq 1$ leads to the following estimate

$$\alpha^2 || T[\beta_1] - T[\beta_2] || < \alpha^2 \frac{\pi^4}{16} \left(\frac{11}{240} + \frac{17}{2400} \pi + \frac{11}{240} 0.278\alpha^2 \right) || \beta_1 - \beta_2 || \\ \doteq \alpha^2 (.4145 + .0776\alpha^2) || \beta_1 - \beta_2 || \quad (h)$$

which is equivalent to (c). The factor of $|| \beta_1 - \beta_2 ||$ in (h) is less than unity provided $\alpha^2 \leq 1.8$. A well-known classical theorem now implies that Eqs. (27), (28) have a *unique* solution so long as $\alpha^2 \leq 1.8$, and that this solution may be obtained by iteration as indicated in (40)*. Moreover, an estimate for the n th iterate is given by (see e.g. [5])

$$|| \beta - \beta_n || \leq \frac{k^n}{1-k} || \beta_1 - \beta_0 ||$$

*The existence of (not necessarily unique) solutions of the basic equations(27), (28) for all values of α^2 is guaranteed by the Schauder fixed point theorem: T is completely continuous and maps the set of functions $|| \beta || \leq \alpha^2$ into the set $|| \beta || \leq .278\alpha^2$, because of inequality (II).

where β is the exact solution. For the initial approximation β_0 we take the solution given in Section 3.

C. Errors due to Numerical Integration. In the process of solving the integral equations (39) numerically by iteration, the integrals are approximated by sums as follows (set $\tilde{\beta} = \beta + y\pi/2$):

$$\int_0^1 \cos \tilde{\beta}(y)G, dy = (1 - x) \int_0^x \cos \tilde{\beta}(y) dy + \int_x^1 (1 - y) \cos \tilde{\beta}(y) dy = h \sum_{i=0}^n c_i I_i + E_1$$

$$\int_0^1 \sin \tilde{\beta}(y)f(y)G_\beta dy = (1 - x) \int_0^x \sin \tilde{\beta}(y)f(y)y dy + x \int_x^1 \sin \tilde{\beta}(y)f(y)(1 - y) dy$$

$$= h \sum_{i=0}^n c_i J_i + E_2 \quad (i)$$

where $h = 1/n$ is the spacing, I_i and J_i are values of the respective integrands evaluated at $y_i = ih$. The quantities E_1, E_2 are bounds for the error whose form depends on the integration formula being used (e.g., see Eq. (43)). Bounds on E_1, E_2 can be found, provided bounds on certain higher derivatives of the integrands are available. It will suffice to illustrate the calculation of such bounds from the results of part A for the special case of approximating the integrals by the trapezoidal rule. From that it will be clear how to do this for Simpson's rule or other integration formulas, such as (43).

The four integrands occurring in (i) are

$$\varphi_1 = \cos \tilde{\beta}(y), \quad \varphi_2 = y\varphi_1, \quad \varphi_3 = [\sin \tilde{\beta}(y)]f(y), \quad \varphi_4 = y\varphi_3$$

Differentiating twice with respect to y , one finds $\varphi_1'' = -\beta'' \sin \tilde{\beta} - (\beta' + \pi/2)^2 \cos \tilde{\beta}$, and similar expressions for $\varphi_2'', \varphi_3'', \varphi_4''$. Using the error formula for the trapezoidal rule, we obtain, with the double bar notation as in Part B:

$$|E_1(x)| \leq \frac{1}{48} h^2 \pi^2 [(1 - x^2) \|\varphi_1''\| + (1 - x) \|\varphi_2''\|] \quad (j)$$

$$|E_2(x)| \leq \frac{1}{48} h^2 \pi^2 \alpha^2 [x(1 - x) \|\varphi_3''\| + 2x(1 - x) \|\varphi_4''\|]$$

In order to evaluate the right hand terms of (j), bounds on $f, f', f'', \beta, \beta', \beta''$ are needed. The bounds (I) and (II) on f and β may be used to obtain bounds on f', β' via the integral expressions given by Eqs. (32), (33), (35), (36) and (38). Bounds on f'', β'' can be obtained directly from the differential equations (7). The results are

$$|f'(x)| \leq \frac{1}{4} \pi^2 x \leq \frac{1}{4} \pi^2, \quad |\beta'(x)| \leq \frac{1}{64} \pi^4 \alpha^2 \left(\frac{11}{12} - x + \frac{x^3}{3} \right) \leq \frac{\pi^4 \alpha^2}{256} \quad (k)$$

$$|f''(x)| \leq \frac{1}{4} \pi^2, \quad |\beta''(x)| \leq \frac{1}{32} \pi^4 \alpha^2 (1 - x^2) \leq \frac{\pi^4 \alpha^2}{32}$$

Introduction of these bounds into the expressions for φ_i'' yields, after some lengthy but simple algebra:

$$\|\varphi_1''\| \leq \frac{1}{32} \pi^4 \alpha^2 + A^2(\alpha), \quad \|\varphi_2''\| \leq \pi + \frac{5}{128} \pi^4 \alpha^2 + A^2(\alpha),$$

$$\|\varphi_3''\| \leq \frac{1}{8} \pi^2 A^2(\alpha) + \frac{1}{2} \pi^2 A(\alpha) + \frac{1}{256} \pi^6 \alpha^2 + \frac{1}{4} \pi^2, \quad (l)$$

$$\|\varphi_4''\| \leq \frac{1}{2} \pi^2 + \frac{1}{4} \pi^2 A(\alpha) + \|\varphi_3''\|; \quad A(\alpha) \equiv \frac{1}{2} \pi \left(1 + \frac{1}{128} \pi^3 \alpha^2 \right).$$

Substitution of (l) into (j) completes the estimation of E_1 and E_2 . A slightly more refined estimate for E_1 can be obtained from

$$|E_1(x)| \leq \frac{1}{48} h^2 \pi^2 [(1-x)x \operatorname{Max}_{0 \leq v \leq x} |\varphi_1''(y)| + (1-x) \operatorname{Max}_{x \leq v \leq 1} (|\varphi_1''(y)| + |\varphi_2''(y)|)]$$

by using the bounds (k) depending on x . Numerical example: For $\alpha = 1$, one obtains from (j), $|E_1| < 3.64 h^2$, $|E_2| < 4.19 h^2$. Hence $h < 0.015$ insures $|E_i| < 10^{-3}$. This is to be compared with the actual $h = 0.05$, used for calculations with $\alpha \leq 1$.

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