A METHOD OF SOLVING A SYSTEM OF LINEAR EQUATIONS WHOSE COEFFICIENTS FORM A TRIDIAGONAL MATRIX*

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1. Introduction. Consider the following system of simultaneous linear equations

$$-2x_1 + x_2 \qquad\qquad\qquad = d_1 ,$$
$$x_1 - 2x_2 + x_3 \qquad\qquad = d_2 ,$$
$$x_2 - 2x_3 + x_4 \qquad\quad = d_3 , \qquad\qquad (1)$$
$$x_3 - 2x_4 + x_5 = d_4 ,$$
$$x_4 - x_5 = d_5 ,$$

where d_i $(i = 1, 2, \cdots , 5)$ are known quantities. Suppose we want to solve Eqs. (1) for different sets of d_i . To be more general, suppose we want to solve the system of equations

$$A\mathbf{x} = \mathbf{d}, \qquad\qquad (2)$$

where A is a non-singular tridiagonal matrix (not necessarily symmetric) whose elements are given and fixed while $\mathbf{d}$ will be changed from time to time. Since A is non-singular, one can obtain the inverse A^{-1} of A and determine $\mathbf{x}$ by

$$\mathbf{x} = A^{-1}\mathbf{d}. \qquad\qquad (3)$$

A^{-1} so obtained, however, is no longer a tridiagonal matrix. If A is a $n \times n$ matrix, we will have in general n^2 elements of A^{-1}. When n is large, this method becomes unwieldy, even with an electronic computer.

Linear equations similar to Eqs. (1) are frequently encountered in problems of mathematical physics. For instance, the backward finite difference method for solving the heat equation requires the solution of equations (2) for each step where A is fixed and $\mathbf{d}$ is changed from step to step. Heat equations also appear in the study of longitudinal impact on visco-plastic rods. The solution of other problems, such as discretely loaded strings and the application of the three moment theorem to continuous beams, result in equations of the form of (2).

Methods are known which enable one to determine $\mathbf{x}$ of (2) more efficiently than by using Eq. (3) (see [1], [2] for example). In the following, we will present another method which is very efficient and convenient for ordinary physical problems. The comparison with the traditional triangular decomposition method is presented in the Appendix.

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2. The method. We shall discuss the following system of equations:

$$b_1 x_1 + a_1' x_2 \qquad\qquad\qquad = d_1 ,$$
$$a_2 x_1 + b_2 x_2 + a_2' x_3 \qquad\qquad = d_2 ,$$
$$a_3 x_2 + b_3 x_3 + a_3' x_4 \qquad = d_3 , \qquad\qquad (4)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$
$$a_{n-1} x_{n-2} + b_{n-1} x_{n-1} + a_{n-1}' x_n = d_{n-1} ,$$
$$a_n x_{n-1} + b_n x_n = d_n .$$

To discuss our method, we will consider first a special case in which $d_1 = d_2 = \cdots = d_{n-1} = 0$ but $d_n \neq 0$. We define:

$$c_2 = -a_1'/b_1 , \qquad c_{i+1} = -a_i'/(a_i c_i + b_i). \qquad\qquad (5)$$

Starting from the first equation of (4) by Gaussian eliminations, we obtain:

$$x_1 = c_2 x_2 ,$$
$$x_2 = c_3 x_3 ,$$
$$\cdots\cdots\cdots , \qquad\qquad (6)$$
$$x_{n-1} = c_n x_n ,$$
$$x_n = d_n/(a_n c_n + b_n).$$

Hence we determine x_n from the last equation of (6) and then determine the other values $x_{n-1} , x_{n-2} , \cdots , x_1$.

Although this is trivial, let us illustrate this special case by taking $d_1 = d_2 = d_3 = d_4 = 0$, $d_5 = -1$ in Eqs. (1). The values of c_i for Eqs. (1) are listed in the second row of Table 1. The last row of Table 1 gives the solution for x_i . Notice that $x_5 = 5$ by the last equation of (6). The other equations of (6) give $x_4 = c_5 x_5 = 4$, $x_3 = c_4 x_4 = 3$ and so on. Hence c_i is the "left carry-over factor" of x_i to x_{i-1} .

<div align="center">

TABLE 1

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$d_1 = d_2 = d_3 = d_4 = 0, d_5 = -1$, Eqs. (1)				
(1) $\quad i = 1$	2	3	4	5
(2) $\quad c_i =$	1/2	2/3	3/4	4/5
(3) $\quad x_i = 1 \leftarrow 2$	$\leftarrow 3$	$\leftarrow 4$	$\leftarrow 5$	

In the same manner, one can introduce the "right carry-over factors". Indeed, when $d_1 \neq 0$ and $d_2 = d_3 = \cdots = d_n = 0$, one obtains

$$x_n = c_{n-1}' x_{n-1} ,$$
$$x_{n-1} = c_{n-2}' x_{n-2}$$
$$\cdots\cdots\cdots\cdots , \qquad\qquad (7)$$
$$x_2 = c_1' x_1 ,$$
$$x_1 = d_1/(a_1' c_1' + b_1),$$

where

$$c'_{n-1} = -a_n/b_n , \qquad c'_{i-1} = -a_i/(a'_i c'_i + b_i). \tag{8}$$

Taking Eqs. (1) as an example again, we put $d_1 = -1$, $d_2 = d_3 = d_4 = d_5 = 0$. The solution for x_i is shown in Table 2.

TABLE 2

$d_1 = -1, d_2 = d_3 = d_4 = d_5 = 0$, Eqs. (1)				
(1) $\quad i = 1$	2	3	4	5
(2) $\quad c'_1 = 1$	1	1	1	1
(3) $\quad x_i = 1 \rightarrow 1$	$\rightarrow 1$	$\rightarrow 1$	$\rightarrow 1$	

Next, we consider the case when d_i are all zero except d_k where $0 < k < n$. The first $(k - 1)$ equations of (4) give:

$$x_1 = c_2 x_2 ,$$

$$x_2 = c_3 x_3 ,$$

$$\cdots\cdots\cdots$$

$$x_{k-1} = c_k x_k .$$

The last $(n - k)$ equations of (4) give:

$$x_n = c'_{n-1} x_{n-1} ,$$

$$x_{n-1} = c'_{n-2} x_{n-2} ,$$

$$\cdots\cdots\cdots\cdots\cdots ,$$

$$x_{k+1} = c'_k x_k .$$

Hence the kth equation of (4) gives:

$$a_k c_k x_k + b_k x_k + a'_k c'_k x_k = d_k .$$

Let us define

$$w_i = 1/(a_i c_i + b_i + a'_i c'_i). \tag{9}$$

Then we have

$$x_k = w_k d_k ;$$

w_i will be called the "weight factor" of x_i . As an illustration, let $d_3 = -1$, $d_1 = d_2 = d_4 = d_5 = 0$ in Eqs. (1). The solution for this case is shown in Table 3. The right carry-over factors c'_i are shown in the second row with the arrow heads ($\rightarrow$) between the factors. Similarly, the left carry-over factors c_i are listed in the fourth row with the arrow heads ($\leftarrow$) between the factors. With these arrow heads, it is easier to identify which carry-over factor is to be used in the calculations.

<div align="center">TABLE 3</div>

$d_1 = d_2 = d_4 = d_5 = 0, d_3 = -1$, Eqs. (1)								
(1)	$i = 1$		2		3		4	5
(2)	$c_i' = 1$	$\rightarrow$	1	$\rightarrow$	1	$\rightarrow$	1	$\rightarrow$
(3)	$w_i =$			-3				
(4)	$c_i =$		$\leftarrow 1/2$	$\leftarrow 2/3$	$\leftarrow 3/4$	$\leftarrow 4/5$		
(5)	$x_i = 1$	$\leftarrow$	2	$\leftarrow$	3	$\rightarrow$	3	$\rightarrow$ 3

Finally, we will illustrate the general case in which d_i are all specified. It is seen from the previous examples that the values of x_i in general consist of three parts, namely: the value due to its own weight, that due to the right carry-over, and that due to the left carry-over. If we denote these values by W_i, C_i' and C_i respectively, we have

$$x_i = W_i + C_i' + C_i \ .$$

To illustrate this, we take $d_1 = d_2 = d_3 = d_4 = d_5 = -1$ in Eqs. (1). The values W_i are obtained by

$$W_i = w_i d_i \ ,$$

where w_i is the weight factor expressed by Eq. (9). Notice that c_1 and c_5' are taken as zero in obtaining w_1 and w_5. The values of w_i and W_i for this case are shown respectively in the third and sixth rows of Table 4. The procedure of calculation may be followed in the table. For instance,

$$C_2' = W_1 c_1' = 1, \qquad C_3' = (W_2 + C_2')c_2' = (2 + 1)\cdot 1 = 3;$$
$$C_4 = W_5 c_5 = 4, \qquad C_2 = (W_3 + C_3)c_3 = (3 + 6)\cdot 2/3 = 6.$$

In other words, C_i' and C_i are obtained by the recurrent formulae

$$C_{i+1}' = (W_i + C_i')c_i'$$
$$C_{i-1} = (W_i + C_i)c_i \ .$$

It should be noticed that C_i' and C_i are obtained independently. One can calculate C_i' first and then C_i or vice versa. W_i, however, must be calculated before C_i' and C_i. Hence, the method has symmetry in the procedure of calculation.

If we are given another set of d_i in Eqs. (1), we can use the same values of c_i', c_i and w_i in Table 4 to calculate W_i, C_i' and C_i. c_i', c_i and w_i are obtained from a_i', a_i and b_i. Once the former are obtained the latter are no longer needed. Notice that the former has exactly the same number of elements as the latter does.

It is seen from Tables 1, 2 and 3 that if there is only one non-zero d_i it is not necessary to calculate all of the carry-over factors and the weight factors to obtain the solution for x_i. On the other hand, if we want to know only one of the x_i, say x_k, we do not need to calculate all of the "carry-over values" C_i' and C_i. Indeed, only those C_i' and C_j where $i \leq k$, $j \geq k$ are needed.

In deriving Eqs. (5), (8) and (9), we tacitly assumed that the denominators do not

<div align="center">TABLE 4</div>

$$d_1 = d_2 = d_3 = d_4 = d_5 = -1, \text{ Eqs. (1)}$$

(1)	$i =$	1	2	3	4	5
(2)	$c_i' =$	1 $\rightarrow$	1 $\rightarrow$	1 $\rightarrow$	1 $\rightarrow$	
(3)	$w_i =$	-1	-2	-3	-4	-5
(4)	$c_i =$		$\leftarrow$ 1/2	$\leftarrow$ 2/3	$\leftarrow$ 3/4	$\leftarrow$ 4/5
(5)	$C_i' =$		1	3	6	10
(6)	$W_i =$	1	2	3	4	5
(7)	$C_i =$	4	6	6	4	
(8)	$x_i =$	5	9	12	14	15

vanish. It can be shown that the denominator of Eq. (9) does not vanish if A is non-singular.

In Eq. (5) we can assume that $b_1 \neq 0$. Should b_1 be zero, one can solve x_2 explicitly from Eqs. (4) and Eqs. (4) are reduced to similar equations with one unknown less. Similarly, we can assume that $b_n \neq 0$ in Eq. (8).

It remains to investigate the possibility that $(a_i c_i + b_i)$ of Eq. (5) and $(a_i' c_i' + b_i)$ of Eq. (8) may vanish, in other words c_i and c_i' may be infinite. We will see later that $c_k = \infty$ implies that the value x_k of Eqs. (4) does not depend on d_k, d_{k+1}, $\cdots$, d_n. Similarly, when $c_k' = \infty$, x_k does not depend on d_1, d_2, $\cdots$, d_k. In many physical problems, for instance in the solution of the heat equation by the backward difference method, the value of x_k depends on all the values d_i. Hence, in most physical problems c_i and c_i' rarely become infinite. For completeness of our method, however, we shall discuss this case in the next section.

3. Modified carry-over factors. As in the first example of section 2, we assume that $d_1 = d_2 = \cdots = d_{n-1} = 0$, $d_n \neq 0$ in Eqs. (4). By Gaussian elimination to the $(k-1)$th equation, one obtains Eqs. (4) in the following form:

$$x_1 = c_2 x_2 ,$$

$$x_2 = c_3 x_3 ,$$

$$\cdots\cdots\cdots ,$$

$$x_{k-2} = c_{k-1} x_{k-1} ,$$

$$(a_{k-1} c_{k-1} + b_{k-1}) x_{k-1} + a_{k-1}' x_k = 0 ,$$

$$a_k x_{k-1} + b_k x_k + a_k' x_{k+1} = 0 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$a_n x_{n-1} + b_n x_n = d_n ;$$

(10)

c_2, c_3, $\cdots$, c_{k-1} are determined by (5) and we assume that they are all finite up to c_{k-1}. Suppose $(a_{k-1} c_{k-1} + b_{k-1}) = 0$ so that c_k does not exist. First we show that $(a_{k-1} c_{k-1} + b_{k-1})$ and a_k cannot both vanish if the matrix A is non-singular. Indeed, if both vanish, the

following set of x_i will satisfy the corresponding homogeneous equations of (10):

$$x_1 = c_2 x_2 , \qquad x_2 = c_3 x_3 , \cdots , x_{k-2} = c_{k-1} x_{k-1} , \tag{11}$$

$$x_k = x_{k+1} = \cdots = x_n = 0.$$

By assuming an arbitrary non-zero value for x_{k-1} in (11) one gets a non-trivial solution of Eqs. (10) with $d_n = 0$. Since Eqs. (10) and Eqs. (4) are equivalent, this implies that Eqs. (4) has a non-trivial solution when d_i are zero. This contradicts the assumption that the matrix A is non-singular. Hence $(a_{k-1} c_{k-1} + b_{k-1})$ and a_k cannot both vanish.

When $(a_{k-1} c_{k-1} + b_{k-1}) = 0$, Eqs. (10) give

$$x_k = 0, \qquad x_{k-1} = -(a'_k/a_k) x_{k+1} .$$

We define:

$$c_{k+1} = 0, \qquad c_{(k+1)} = -a'_k/a_k \quad when \quad c_k = \infty .$$

Then we have

$$x_k = c_{k+1} x_{k+1} = 0, \qquad x_{k-1} = c_{(k+1)} x_{k+1} .$$

$c_{(k+1)}$ will be called the "modified left carry-over factor" of x_{k+1} . It defines the portion of x_{k+1} to be carried over to x_{k-1} instead of x_k . Since we have proved that $a_k \neq 0$, $c_{(k+1)}$ exists.

The procedure of deriving the modified carry-over factors is equivalent to interchanging two equations to avoid a zero pivot.

It should be noticed from Eqs. (10) that when $(a_{k-1} c_{k-1} + b_{k-1}) = 0$, i.e., $c_k = \infty$, x_k is zero even if d_k , d_{k+1} , $\cdots$, d_n are non-zero. Hence x_k is independent of d_k , d_{k+1} , $\cdots$, d_n when $c_k = \infty$.

By decomposing A into a lower triangular matrix and an unit upper triangular matrix (see Eq. (22) of the Appendix), one can show that $(a_{k-1} c_{k-1} + b_{k-1}) = 0$ implies that the $(k-1)$th leading principal minor of A,

$$\begin{vmatrix} b_1 & a'_1 & & & \\ a_2 & b_2 & a'_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{k-2} & b_{k-2} & a'_{k-2} \\ & & & a_{k-1} & b_{k-1} \end{vmatrix} = 0.$$

Hence, $(a_i c_i + b_i) = 0$ if and only if a leading principal minor of A vanishes. If A is symmetric positive definite (which is the case in many physical problems), no minors are zero [3].

As an example, consider the following system of equations:

$$-2x_1 + x_2 \qquad\qquad\qquad = d_1 ,$$

$$2x_1 - x_2 + x_3 \qquad\qquad = d_2 ,$$

$$x_2 - 2x_3 + x_4 \qquad = d_3 , \tag{12}$$

$$x_3 - 2x_4 + x_5 = d_4 ,$$

$$x_4 - x_5 = d_5 .$$

The left carry-over factors c_i are listed in the fourth row of Table 5. Here we have $c_2 = \frac{1}{2}$ and $c_3 = \infty$. Hence $c_4 = 0$ and $c_{(4)} = -1$ by our definition. In Table 5, we do not list c_3 and c_4. In the place of c_4, we put $c_{(4)}$. The long arrow head between c_2 and $c_{(4)}$ suggests that the carry-over from $i = 4$ should skip $i = 3$ and go to $i = 2$. There is no carry-over to $i = 3$. In the fifth row we give a special case in which $d_1 = d_2 = d_3 = d_4 = 0$, $d_5 = -2$. The solution for this case is shown in the sixth row. $x_3 = 0$ in this case.

A similar definition can be applied to the modified right carry-over factors. We define: *when $c'_k = \infty$, $c'_{k-1} = 0$ and $x_{k+1} = c'_{(k-1)}x_{k-1}$ where $c'_{(k-1)} = -a_k/a'_k$*. The right carry-over factors c'_i for Eqs. (12) are shown in the second row of Table 5. Here we have $c'_1 = \infty$. Since there is no c'_0, we can ignore the modified carry-over factor $c'_{(0)}$.

TABLE 5. *Solutions of Eqs. (12)*

(1)	$i =$	1	2	3	4	5
(2)	$c'_i =$	$\infty \longrightarrow$	$1 \longrightarrow$	$1 \longrightarrow$	$1 \longrightarrow$	
(3)	$w_i =$	$\overrightarrow{(1)}$	1	$\overleftarrow{(1)}$	-1	-2
(4)	$c_i =$		$\longleftarrow 1/2 \longleftarrow$		$(-1) \longleftarrow$	$1/2$
(5)	$d_i =$	0	0	0	0	-2
(6)	$x_i =$	$-1 \longleftarrow$	$-2 \longleftarrow$		$2 \longleftarrow$	4
(7)	$d_i =$	1	0	0	0	0
(8)	$x_i =$	$0 \dashrightarrow$	$1 \longrightarrow$	$1 \longrightarrow$	$1 \longrightarrow$	1
(9)	$d_i =$	0	0	2	0	0
(10)	$x_i =$	$1 \longleftarrow$	$2 \longleftarrow$	$0 \longrightarrow$	$0 \longrightarrow$	0
(11)	$d_i =$	1	2	2	2	-2
(12)	$C'_i =$		1	3	3	1
(13)	$W_i =$	0	2	0	-2	4
(14)	$C_i =$	2	2 $\quad$ 0		2	
(15)	$x_i =$	2	5	3	3	5

Finally, we consider the case in which d_i are all zero except d_k, where $0 < k < n$. By Gaussian eliminations from the first and the last of Eqs. (4) respectively, the $(k-1)$th, kth and $(k+1)$th equations of (4) can be written as:

$$(a_{k-1}c_{k-1} + b_{k-1})x_{k-1} + a'_{k-1}x_k = 0,$$
$$a_k x_{k-1} + b_k x_k + a'_k x_{k+1} = d_k, \qquad (13)$$
$$a_{k+1}x_k + (b_{k+1} + a'_{k+1}c'_{k+1})x_{k+1} = 0.$$

If c_k, c'_k both exist, the second equation of (13) yields $x_k = w_k d_k$ as discussed in Eq. (9). We shall investigate the case in which c_k or c'_k is infinite, i.e., the case in which $(a_{k-1}c_{k-1} + b_{k-1})$ or $(b_{k+1} + a'_{k+1}c'_{k+1})$ vanishes.

First, one can prove that $(a_{k-1}c_{k-1} + b_{k-1})$ and $(b_{k+1} + a'_{k+1}c'_{k+1})$ cannot both vanish if A is non-singular. The proof is similar to the one we proved in the beginning of this

section. Should they both vanish, one can find a non-trivial solution of Eqs. (13) with $d_k = 0$ and hence a non-trivial solution of Eqs. (4) with all $d_i = 0$. This leads to a contradiction.

Suppose $(a_{k-1}c_{k-1} + b_{k-1}) = 0$ while $(b_{k+1} + a'_{k+1}c'_{k+1}) \neq 0$. Eqs. (13) give:

$$x_k = 0, \qquad x_{k+1} = 0, \quad \text{and} \quad x_{k-1} = d_k/a_k .$$

We define:

When $c_k = \infty$, $w_k = 0$ and $x_{k-1} = w_{(k)}d_k$ where $w_{(k)} = 1/a_k$. $w_{(k)}$ is the modified weight factor which defines the weight factor by which d_k should be "carried-over" to x_{k-1} instead of "imposed" on x_k. The fact that $(a_{k-1}c_{k-1} + b_{k-1})$ and a_k cannot both vanish assures us the existence of $w_{(k)}$.

To illustrate the use of $w_{(k)}$, we take $d_1 = d_2 = d_4 = d_5 = 0$, $d_3 = 2$ in Eqs. (12). This is shown in the ninth row of Table 5. Since $c_3 = \infty$, $w_3 = 0$ and $w_{(3)} = 1$ by our definition. In the third row, in the place of w_3, we put $w_{(3)}$. The arrow head on the top of $w_{(3)}$ suggests that the "weight" of d_3 should be carried over to x_2 which is to the "left" of x_3. Now, since $w_3 = 0$, $x_3 = w_3d_3 = 0$ and $x_2 = w_{(3)}d_3 = 2$. $x_4 = c'_3x_3 = 0$, $x_5 = c'_4x_4 = 0$ and $x_1 = c_2x_2 = 1$. This is shown in the tenth row.

The counterpart of $w_{(k)}$ is $w'_{(k)}$, which is defined in the same manner:

When $c'_k = \infty$, $w_k = 0$ and $x_{k+1} = w'_{(k)}d_k$ where $w'_{(k)} = 1/a'_k$.

In Eqs. (12), $c'_1 = \infty$. Hence $w_1 = 0$ and $w'_{(1)} = 1$. This is shown in the third row of Table 5. The solution of Eqs. (12) when d_i are specified in the seventh row is shown in the eighth row.

In closing this section we give an example in which d_i of Eqs. (12) are all specified as indicated in the eleventh row of Table 5. The procedure of calculation is the same as in Table 4 except that whenever $w_k = 0$, the values $w_{(k)}d_k$ or $w'_{(k)}d_k$ are not placed in the row of W_i but carried over to the row C_i or C'_i. $w_{(k)}d_k$ and $w'_{(k)}d_k$ are easily identified in Table 5 by the dotted-lined arrow heads.

4. An application. The method is also applicable to problems such as that of heat conduction in a semi-infinite rod, where the difference equations involve an "infinite" number of dependent variables. To illustrate, we consider the problem represented as follows:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = 0,$$

$$v(x, 0) = 0, \qquad 0 < x < \infty, \tag{14}$$

$$v(0, t) = 1, \qquad 0 \le t.$$

The solution of (14) is given by

$$v(x, t) = \text{erfc}\left(\frac{x}{2\sqrt{t}}\right), \tag{15}$$

where

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} \, du.$$

Let us solve Eqs. (14) by the backward finite difference approximation. By choosing the mesh ratio $\Delta t/(\Delta x)^2 = 1$, and denoting $v(i\Delta x, j\Delta t)$ by $v_{i,j}$, one obtains the following "infinite" system of equations:

$$
\begin{aligned}
v_{1,j} &= 1, \\
-v_{1,j} + 3v_{2,j} - v_{3,j} &= v_{2,j-1}, \\
-v_{2,j} + 3v_{3,j} - v_{4,j} &= v_{3,j-1}, \\
-v_{3,j} + 3v_{4,j} - v_{5,j} &= v_{4,j-1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
\tag{16}
$$

The left carry-over factors are

$$
c_2 = 0, \qquad c_{i+1} = 1/(3 - c_i). \tag{17}
$$

This is shown in the fourth row of Table 6. Notice that c_i approaches a constant value as i increases. In fact, the difference equation (17) has a closed form solution (see [4]).

$$
c_{i+2} = \frac{\sinh \alpha i}{\sinh \alpha(i + 1)}, \quad \text{where} \quad e^\alpha = (3 + \sqrt{5})/2. \tag{18}
$$

Hence the asymptotic value of c_i as $i \to \infty$ is

$$
c_\infty = e^{-\alpha} = (3 - \sqrt{5})/2 = 0.38196 \cdots. \tag{19}
$$

Now, let us determine the right carry-over factors. We can assume that $v(\infty, t) = 0$. Then we have

$$
c'_\infty = 0, \qquad c'_{i-1} = 1/(3 - c'_i). \tag{20}
$$

This is a difference equation similar to Eq. (17). According to the asymptotic behavior discussed above, we conclude that

$$
c'_1 = c'_2 = \cdots = c'_i = (3 - \sqrt{5})/2 = 0.38196 \cdots,
$$

as long as i is finite. This is listed in the second row of Table 6. The third row then gives the weight factors w_i. All the numbers in Table 6 are rounded off to three digits after the decimal point.

We remark that the asymptotic solution c_∞ of Eq. (17) is independent of the initial value c_2. Hence, the assumption that $c'_\infty = 0$ in Eq. (20) is immaterial.

Since $v_{1,0} = 1$, and $v_{i,0} = 0$, $i \neq 1$, the solution for the first step $v_{i,1}$ is readily obtained by using the factors c'_i alone. This is shown in the fifth row. The solutions for the second and third steps are shown respectively in the ninth and thirteenth rows. In the eighth and twelfth rows, we have assumed that $C_{8,2}$ and $C_{8,3}$ are small and negligible. For comparison, we list the solution $v(i\Delta x, 3\Delta t)$ obtained from Eq. (15) in the last row. It should be noticed that while $v(i\Delta x, 3\Delta t)$ are the exact solution of Eqs. (14), $v_{i,3}$ are the "exact" solutions (with ± 1 round-off error in the last digit) of Eqs. (16).

5. Conclusions. The method described here is very convenient and efficient for most physical problems which have the system of equations in the form of Eqs. (4). The method has symmetry in the procedure of calculation and has flexibility in that one can eliminate unnecessary computations. For instance, if we want to determine a particular x_i only, say x_k, we do not need to find all or part of x_i before we know the value of x_k. We can determine x_k independently.

TABLE 6. *Solutions of Eqs. (16)*

(1)	$i =$	1	2	3	4	5	6	7	8 ···
(2)	$c'_i =$	.382 →	.382 →	.382 →	.382 →	.382 →	.382 →	.382 →	.382 ··
(3)	$w_i =$	1.000	.382	.438	.446	.447	.448	.448	.448 ··
(4)	$c_i =$		← 0	← .333	← .375	← .381	← .382	← .382	← .382 ··
(5)	$v_{i,1} =$	1.000	.382	.146	.056	.021	.008	.003	.001 ··
(6)	$C'_{i,2} =$		.382	.202	.102	.049	.022	.010	.004 ··
(7)	$W_{i,2} =$	1.000	.146	.064	.025	.009	.004	.001	.000 ··
(8)	$C_{i,2} =$	0	.025	.011	.004	.002	.000	.000	— ··
(9)	$v_{i,2} =$	1.000	.553	.277	.131	.060	.026	.011	.004 ··
(10)	$C'_{i,3} =$		.382	.226	.133	.073	.038	.019	.009 ·
(11)	$W_{i,3} =$	1.000	.211	.121	.058	.027	.012	.005	.002 ··
(12)	$C_{i,3} =$	0	.049	.026	.012	.005	.002	.001	— ··
(13)	$v_{i,3} =$	1.000	.642	.373	.203	.105	.052	.025	.011 ··
	$v(i\Delta x, 3\Delta t) =$	1.000	.683	.414	.220	.102	.040	.014	.004 ··

When the carry-over factor is infinite, a modified method is offered. This, perhaps, may complicate the programming for use in an electronic computer. However, this is rarely needed in ordinary physical problems.

Theoretically, this method is applicable regardless of the magnitude of the carry-over factors. In numerical calculations, however, a large carry-over factor can cause serious round-off errors in the solution. Another "modified" method could be devised to prevent the large factors if one followed the classical method of selecting the largest pivot.

APPENDIX

We shall discuss briefly the method described in section 2 in matrix notation and compare it with the traditional method of triangular decomposition. Let us write Eqs. (4) as

$$A\mathbf{x} = \mathbf{d}. \tag{21}$$

According to our method, the matrix A is first decomposed into a lower triangular matrix L and an unit upper triangular matrix C, namely,

$$A = LC \tag{22}$$

where

$$L = \begin{bmatrix} b_1 & & & \\ a_2 & b_2 + c_2 a_2 & & \\ & \cdot & \cdot & \\ & & a_n & b_n + c_n a_n \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -c_2 & & & \\ & 1 & -c_3 & & \\ & & \cdot & \cdot & \\ & & & 1 & -c_n \\ & & & & 1 \end{bmatrix}.$$

c_i are defined in Eqs. (5). Next, in the same manner, we decompose A into an upper triangular matrix U and an unit lower triangular matrix C', namely,

$$A = UC' \tag{23}$$

where

$$U = \begin{bmatrix} b_1 + c_1'a_1' & & a_1' & \\ & \ddots & & \ddots \\ & & b_{n-1} + c_{n-1}'a_{n-1}' & a_{n-1}' \\ & & & b_n \end{bmatrix}, \quad C' = \begin{bmatrix} 1 & & & \\ -c_1' & \cdot & 1 & \\ & \cdot & \cdot & \\ & & -c_{n-2}' & \cdot & 1 \\ & & & -c_{n-1}' & 1 \end{bmatrix}.$$

c_i' are defined in Eqs. (8). If we define the diagonal matrix W by

$$W^{-1} = \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_n \end{bmatrix}$$

where w_i are expressed by Eqs. (9), we have, finally

$$A = L + U - W. \tag{24}$$

Equation (24) with Eqs. (22) and (23) give

$$W = L + U - A$$
$$= AC^{-1} + AC'^{-1} - A$$
$$= A(C^{-1} + C'^{-1} - I).$$

Hence

$$A^{-1} = (C^{-1} + C'^{-1} - I)W^{-1}$$

and the solution of Eq. (21) becomes

$$\mathbf{x} = (C^{-1} + C'^{-1} - I)(W^{-1}\mathbf{d}). \tag{25}$$

This is the method described in section 2 (see Table 4).

The traditional method is equivalent to decomposing A into a lower triangular matrix L and an unit upper triangular matrix C as shown in Eq. (22), and then decomposing L into a diagonal matrix D and an unit lower triangular matrix C^*. In other words,

$$A = LC = DC^*C \tag{26}$$

where

$$D = \begin{bmatrix} b_1 & & & \\ & b_2 + c_2a_2 & & \\ & & \ddots & \\ & & & b_n + c_na_n \end{bmatrix}, \quad C^* = \begin{bmatrix} 1 & & & \\ \dfrac{a_n}{b_2 + c_2a_2} & 1 & & \\ & \ddots & \ddots & \\ & & \dfrac{a_n}{b_n + c_na_n} & 1 \end{bmatrix}.$$

Hence

$$A^{-1} = C^{-1}C^{*-1}D^{-1}$$

and the solution of (21) is

$$\mathbf{x} = C^{-1}C^{*-1}(D^{-1}\mathbf{d}) \qquad (27)$$

In the traditional method, one can treat the elements of D as weight factors and the co-diagonal elements of C and C^* as carry-over factors. The procedure of calculation would be slightly different from our method. If one wanted to determine a particular element of $\mathbf{x}$ only, one has to invert the whole matrix C^* and part of the matrix C by Eq. (27), whereas by Eq. (25) only parts of matrices C and C' need to be inverted.

Should one of the diagonal elements of the matrix L vanish, the corresponding co-diagonal elements of C and C^* become infinite and the methods represented by Eqs. (25) and (27) fail. Matrix notation for the method using modified carry-over factors are omitted here. However, if one follows the method described in section 3 and writes the method in matrix notation, one will find the modified carry-over factor placed in the third diagonal of C (or C') with two zeros in the co-diagonal adjacent to it. There is a corresponding non-zero element in the third diagonal of L (or U).

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