SEPARATION AND INTERLACING THEOREMS*

By

LOUIS C. BARRETT AND GERALD E. BENDIXEN

South Dakota School of Mines and Technology and Naval Ordnance Test Station, China Lake, California

1. Introduction. A difficulty which is frequently encountered, when one undertakes to solve a boundary value problem, is that of finding the related characteristic numbers. In many instances, it proves impracticable, if not impossible, to determine explicitly these numbers by means of any known formula. One must therefore resort to numerical methods for their computation. For this purpose, it is desirable to have some procedure for isolating one characteristic number from all others. This study contains separation and interlacing theorems which serve to fill this need. Indeed, they do even more than this; for, they provide information concerning separation and interlacing properties of the zeros of functions $f_i(\lambda)$ and $F(\lambda)$ in an equation of the type

$$F(\lambda) \equiv f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda) = 0. \tag{1}$$

To attribute a precise meaning to the concepts with which we shall be primarily concerned, let us begin by introducing the following definitions concerning two functions $f(\lambda)$ and $g(\lambda)$.

Definition 1. It will be said that the zeros of $f(\lambda)$ *separate* the zeros of $g(\lambda)$ on an interval (Λ_1, Λ_2) provided that on this interval

(*i*) $f(\lambda)$ and $g(\lambda)$ have no common zero,
(*ii*) all zeros of $f(\lambda)$ and $g(\lambda)$ are simple, and
(*iii*) between any two consecutive zeros of $g(\lambda)$ there lies exactly one zero of $f(\lambda)$.

From this definition, it is clear that when the zeros of $f(\lambda)$ separate those of $g(\lambda)$ the converse is also true on each subinterval (g_0, G_0) of (Λ_1, Λ_2) whose end points g_0 and G_0 are zeros of $g(\lambda)$.

Definition 2. It will be said that the zeros of $f(\lambda)$ *interlace* the zeros of $g(\lambda)$ on an interval (Λ_1, Λ_2) provided that on this interval

(*i*) no simple zero of $g(\lambda)$ is a zero of $f(\lambda)$,
(*ii*) $f(\lambda)$ has a zero equal to each double zero of $g(\lambda)$,
(*iii*) all zeros of $f(\lambda)$ are simple whereas the zeros of $g(\lambda)$ are either simple or double, and
(*iv*) between any two consecutive zeros of $g(\lambda)$, whether simple or double, there lies exactly one zero of $f(\lambda)$.

Observe that Definition 1 is, in reality, just a special case of Definition 2, corresponding to functions $g(\lambda)$ which have no double zeros. Thus, if on an interval (Λ_1, Λ_2) the zeros of $f(\lambda)$ separate the zeros of $g(\lambda)$, then the zeros of $f(\lambda)$ also interlace the zeros of $g(\lambda)$. The preceding statement, in general, becomes false when the words *separate* and *interlace* are interchanged.

*Received April 15, 1964; revised manuscript received July 10, 1964.

2. Lemmas pertaining to zeros of continuous functions. We next state and prove three lemmas which will be used in our subsequent work.

LEMMA 1. Let the functions $f(\lambda)$ and $g(\lambda)$ be continuous, together with their first two derivatives, when $\Lambda_1 < \lambda < \Lambda_2$. For each such value of λ suppose, in addition, that

$$W(\lambda) = \begin{vmatrix} f'(\lambda) & g'(\lambda) \\ f(\lambda) & g(\lambda) \end{vmatrix} \neq 0.$$

Then, on (Λ_1, Λ_2), the zeros of $f(\lambda)$ separate the zeros of $g(\lambda)$, and conversely.

We shall prove the lemma by showing that $f(\lambda)$ and $g(\lambda)$ satisfy the three conditions of Definition 1. We shall also use the following definition.

DEFINITION 3. A function $\Phi(\lambda)$ will be said to have a zero of order k at $\lambda = \lambda_0$, if and only if,

$$\Phi(\lambda) \equiv (\lambda - \lambda_0)^k \phi(\lambda),$$

where $k > 0$ and $\phi(\lambda_0) \neq 0$.

Proof of Lemma 1. On (Λ_1, Λ_2), $f(\lambda)$ and $g(\lambda)$ have no equal zeros, because any zero of both $f(\lambda)$ and $g(\lambda)$ would be a zero of $W(\lambda)$ as well. The first condition of Definition 1 is therefore satisfied.

The second condition is also fulfilled, i.e., $f(\lambda)$ and $g(\lambda)$ have at most simple zeros on (Λ_1, Λ_2). For if either function should have a zero of order $k > 1$ on this interval, $W(\lambda)$ would vanish there by Definition 3. On the other hand, the required continuity conditions would be violated if either function were to have a zero of order $k < 1$.

It remains to be shown that between any two consecutive zeros of $g(\lambda)$ on (Λ_1, Λ_2) there lies exactly one zero of $f(\lambda)$, and conversely. This will be done with the aid of Sturm's separation theorem [1].

Now, it is a routine matter to verify that on $\Lambda_1 < \lambda < \Lambda_2$, $f(\lambda)$ and $g(\lambda)$ are solutions of the differential equation

$$\frac{d}{d\lambda}[K(\lambda)y'(\lambda)] + G(\lambda)y(\lambda) = 0, \tag{2}$$

where

$$K(\lambda) \equiv W^{-1}(\lambda) \quad \text{and} \quad G(\lambda) \equiv [f''(\lambda)g'(\lambda) - g''(\lambda)f'(\lambda)]W^{-2}(\lambda)$$

From the hypotheses of the lemma, which require that $f''(\lambda)$ and $g''(\lambda)$ shall be continuous, while $W(\lambda) \neq 0$, it follows that on $\Lambda_1 < \lambda < \Lambda_2$,

(*i*) $K(\lambda)$ does not vanish and $K'(\lambda)$ is continuous,
(*ii*) $G(\lambda)$ is continuous, and
(*iii*) $f(\lambda)$ and $g(\lambda)$ are linearly independent solutions of (2).

In writing the third of these statements, we have recognized that $W(\lambda)$ is simply the Wronskian of $f(\lambda)$ and $g(\lambda)$.

We may now invoke the Sturm separation theorem to conclude that between each pair of consecutive zeros of $g(\lambda)$ on (Λ_1, Λ_2) there is exactly one zero of $f(\lambda)$, and conversely. This completes the proof of Lemma 1.

LEMMA 2. Let $f'(\lambda)$ exist at each point of a closed interval having λ_s and λ_t as end points. Suppose also that $f(\lambda_s) \neq 0$, $f'(\lambda_t) \neq 0$, $f(\lambda_t) = 0$, and $f(\lambda_s)/f'(\lambda_t) > 0$ (< 0); $\lambda_s < \lambda_t$ $(\lambda_t < \lambda_s)$. Then, $f(\lambda)$ has at least one zero between λ_s and λ_t.

It will suffice to prove the lemma in the first instance only, i.e., when

$$f(\lambda_s)/f'(\lambda_t) > 0 \quad \text{and} \quad \lambda_s < \lambda_t\,,$$

because the other case can be handled in a similar manner.

Proof of Lemma 2. If $\lambda_s < \lambda_t$ and $f'(\lambda_t) > 0$ (< 0), then $f(\lambda_s) > 0$ (< 0). Moreover, from the definition of a derivative, there exists an h such that $0 < h < \lambda_t - \lambda_s$ and $f(\lambda_t - h) < 0$ (> 0). Hence, $f(\lambda_s)f(\lambda_t - h) < 0$, i.e., the continuous function $f(\lambda)$ changes sign on $\lambda_s < \lambda < \lambda_t$ and, therefore, has at least one zero between λ_s and λ_t.

LEMMA 3. Let $f'(\lambda)$ exist at each point of a closed interval having λ_s and λ_t as end points. Suppose also that $f(\lambda_s) = f(\lambda_t) = 0$ and $f'(\lambda_s)f'(\lambda_t) > 0$. Then $f(\lambda)$ has at least one zero between λ_s and λ_t.

Because the hypotheses are symmetric in λ_s and λ_t, no loss in generality results by assuming that $\lambda_s < \lambda_t$.

Proof of Lemma 3. If $f'(\lambda_s) > 0$ (< 0), then $f'(\lambda_t) > 0$ (< 0). From the definition of a derivative, there exists an h such that $0 < h < (\lambda_t - \lambda_s)/2$, $f(\lambda_s + h) > 0$ (< 0), and $f(\lambda_t - h) < 0$ (> 0). Since $f(\lambda_s + h)f(\lambda_t - h) < 0$, where $\lambda_s + h < \lambda_t - h$, the continuous function $f(\lambda)$ changes sign on $\lambda_s < \lambda < \lambda_t$ and, therefore, has at least one zero between λ_s and λ_t.

3. Statements and proofs of the separation and interlacing theorems. For $i, j = 1, 2, 3, 4$, define

$$W_{ij}(\lambda) \equiv \begin{vmatrix} f_i'(\lambda) & f_j'(\lambda) \\ f_i(\lambda) & f_j(\lambda) \end{vmatrix}, \tag{3}$$

$$w_{ij}(\lambda) \equiv \begin{vmatrix} F'(\lambda) & [f_i(\lambda)f_j(\lambda)]' \\ F(\lambda) & f_i(\lambda)f_j(\lambda) \end{vmatrix}, \tag{4}$$

and suppose that, at each point λ of the interval $\Lambda_1 < \lambda < \Lambda_2$, the following conditions are satisfied:

(H_1) $W_{12}(\lambda)W_{34}(\lambda) < 0$,

(H_2) each of the functions $f_i(\lambda)$ has a continuous derivative of order two, $i = 1, 2, 3, 4$.

Then the separation and interlacing properties, stated in the following four theorems, apply to the zeros of the functions $F(\lambda)$ and $f_i(\lambda)$ of Eq. (1).

THEOREM 1. On (Λ_1, Λ_2) the zeros of $f_1(\lambda)$ separate the zeros of $f_2(\lambda)$, and conversely.

THEOREM 2. On (Λ_1, Λ_2) the zeros of $f_3(\lambda)$ separate the zeros of $f_4(\lambda)$, and conversely.

THEOREM 3. On (Λ_1, Λ_2) the zeros of $F(\lambda)$ interlace the zeros of the product function $f_1(\lambda)f_3(\lambda)$.

THEOREM 4. On (Λ_1, Λ_2) the zeros of $F(\lambda)$ interlace the zeros of the product function $f_2(\lambda)f_4(\lambda)$.

From Theorems 3 and 4 it is readily seen how the zeros of $f_1(\lambda)$ and $f_3(\lambda)$ [or the zeros of $f_2(\lambda)$ and $f_4(\lambda)$] may be utilized to isolate the zeros of $F(\lambda)$ whenever hypotheses (H_1) and (H_2) are complied with.

We now proceed with proofs of the foregoing theorems.

Proof of Theorems 1 *and* 2. Condition (H_1) guarantees that on $\Lambda_1 < \lambda < \Lambda_2$

$$W_{12}(\lambda) \neq 0 \quad \text{and} \quad W_{34}(\lambda) \neq 0. \tag{5}$$

Hence, in view of the continuity conditions (H_2), Theorems 1 and 2 may be identified as particular instances of Lemma 1.

As for Theorems 3 and 4, their hypotheses are identical, so that we need prove only Theorem 3. Our method of proof will be to show that the four conditions of Definition 2 are satisfied when $f(\lambda)$ and $g(\lambda)$ of that definition are replaced by $F(\lambda)$ and $f_1(\lambda)f_3(\lambda)$, respectively. Throughout the proof, it will be well to bear in mind that Theorems 1 and 2 imply that the functions $f_i(\lambda)$ have at most simple zeros on (Λ_1, Λ_2).

Proof of Theorem 3. Condition (*i*) of Definition 2 is satisfied. For, suppose λ_0 is a simple zero of $f_1(\lambda)f_3(\lambda)$. Then, either $f_1(\lambda_0) = 0$, or else $f_3(\lambda_0) = 0$, but not both. If $f_1(\lambda_0) = 0$, then from Theorem 1, $f_2(\lambda_0) \neq 0$; hence

$$F(\lambda_0) = -f_2(\lambda_0)f_3(\lambda_0) \neq 0.$$

On the other hand, if $f_3(\lambda_0) = 0$, then from Theorem 2, $f_4(\lambda_0) \neq 0$, and

$$F(\lambda_0) = f_1(\lambda_0)f_4(\lambda_0) \neq 0.$$

Therefore, no simple zero of $f_1(\lambda)f_3(\lambda)$ is a zero of $F(\lambda)$.

Condition (*ii*) of Definition 2 also holds, that is to say, $F(\lambda)$ has a zero equal to each double zero of $f_1(\lambda)f_3(\lambda)$. For, if λ_0 is a double zero of $f_1(\lambda)f_3(\lambda)$, then $f_1(\lambda_0) = f_3(\lambda_0) = 0$ so that $F(\lambda_0) = 0$ as well.

We next note that, in as much as $f_1(\lambda)$ and $f_3(\lambda)$ have at most simple zeros on (Λ_1, Λ_2), the zeros of $f_1(\lambda)f_3(\lambda)$ on this interval must be either simple or double. Thus, part of condition (*iii*) of Definition 2 is satisfied. The rest of (*iii*), and condition (*iv*) besides, will be satisfied provided that on $\Lambda_1 < \lambda < \Lambda_2$:

(C_1) all zeros of $F(\lambda)$ are simple;
(C_2) between any two consecutive zeros of $f_1(\lambda)\cdot f_3(\lambda)$, whether simple or double, there lies exactly one zero of $F(\lambda)$.

We shall find it helpful to have the following identities available. They may be derived from (4), together with (1) and (3).

$$w_{13}(\lambda) \equiv f_3^2(\lambda)W_{12}(\lambda) - f_1^2(\lambda)W_{34}(\lambda), \tag{6a}$$

$$w_{24}(\lambda) \equiv f_4^2(\lambda)W_{12}(\lambda) - f_2^2(\lambda)W_{34}(\lambda). \tag{6b}$$

These results, along with (H_1) and (H_2), clearly demonstrate that on (Λ_1, Λ_2):

(D_1) $w_{13}(\lambda)w_{24}(\lambda) \geqq 0$; moreover, $w_{13}(\lambda)$ and $w_{24}(\lambda)$ do not change sign;
(D_2) $w_{13}(\lambda) = 0$ at a point λ_0 if, and only if, λ_0 is a double zero of $f_1(\lambda)f_3(\lambda)$;
(D_3) $w_{24}(\lambda) \neq 0$ except at a double zero of $f_2(\lambda)f_4(\lambda)$;
(D_4) $F(\lambda)$, $f_1(\lambda)f_3(\lambda)$, and $f_2(\lambda)f_4(\lambda)$ have continuous derivatives of order two;
(D_5) the functions defined by (3) and (4) have continuous first order derivatives.

Now, there are three essentially distinct possibilities with regard to any two *consecutive* zeros λ_s and λ_t of $f_1(\lambda)f_3(\lambda)$ on (Λ_1, Λ_2):

Case I: both λ_s and λ_t are simple zeros of $f_1(\lambda)f_3(\lambda)$;

Case II: λ_s is a simple zero and λ_t a double zero of $f_1(\lambda)f_3(\lambda)$;
Case III: λ_s and λ_t are both double zeros of $f_1(\lambda)f_3(\lambda)$.

We proceed to establish (C_1) and (C_2) under each of these three circumstances.

Case I: In this instance, (D_2) shows that $w_{13}(\lambda) \neq 0$ on the closed interval I having λ_s and λ_t as end points. But, according to (D_5), $w_{13}(\lambda)$ is continuous on (Λ_1, Λ_2). Hence, there exists an open interval, having all points of I as interior points, over which $w_{13}(\lambda) \neq 0$. From (D_4) and Lemma 1, it follows that on this open interval the zeros of $F(\lambda)$ separate the zeros of $f_1(\lambda)f_3(\lambda)$. An immediate extension of this conclusion is that

(c_1) between each pair of consecutive simple zeros of $f_1(\lambda)f_3(\lambda)$ on (Λ_1, Λ_2) there is precisely one zero of $F(\lambda)$, and it is of order one.

Case II: By means of (1), (4), (D_1), (D_2), and (D_3), it is easy to verify, in this case, that

$$w_{13}(\lambda_s)w_{24}(\lambda_t) = -P(\lambda_s, \lambda_t)F(\lambda_s)F'(\lambda_t) > 0, \tag{7}$$

where

$$P(\lambda_s, \lambda_t) = f_2(\lambda_t)f_4(\lambda_t)[f_1(\lambda)f_3(\lambda)]'_{\lambda=\lambda_s} \tag{8}$$

It has already been pointed out that λ_t is a zero of $F(\lambda)$. It it were a zero of order $k < 1$, then $F'(\lambda_t)$ would not exist contradicting (D_4). Were it a zero of order $k > 1$, then $F'(\lambda_t)$, and perforce $w_{13}(\lambda_s)w_{24}(\lambda_t)$, would vanish contradicting (7). Hence, λ_t is a simple zero of $F(\lambda)$. But, this implies the more general result concerning zeros of $f_1(\lambda)f_3(\lambda)$, lying on (Λ_1, Λ_2):

(c_2) a double zero of $f_1(\lambda)f_3(\lambda)$, immediately preceded or succeeded by a simple zero of the function, is a simple zero of $F(\lambda)$.

Our next objective will be to show that:

(c_3) between any two consecutive zeros of $f_1(\lambda)f_3(\lambda)$ on (Λ_1, Λ_2), one of which is simple and the other double, there lies exactly one zero of $F(\lambda)$, and it is of order one.

To accomplish this, we first note that any zero $F(\lambda)$ may have between λ_s and λ_t must be of order one. For, if the point λ_t is omitted from the closed interval having λ_s and λ_t as end points, a half open interval is obtained over which $w_{13}(\lambda) \neq 0$, in accordance with (D_2). Thus, either (D_2) or (D_4) would be violated if $F(\lambda)$ had other than a simple zero between λ_s and λ_t.

As a matter of fact, $F(\lambda)$ can have at most one zero between λ_s and λ_t. For, otherwise, Lemma 1 would assure the existence of a zero for $f_1(\lambda)f_3(\lambda)$ between λ_s and λ_t, which is impossible. Therefore, (c_3) will be established as soon as $F(\lambda)$ is shown to have at least one zero between λ_s and λ_t. In proving this, we shall need several inequalities not as yet set down.

Since the zeros of $f_1(\lambda)$ and $f_3(\lambda)$ are all simple, we are assured that $f_1'(\lambda_t) \neq 0$ and $f_3'(\lambda_t) \neq 0$. Now λ_s is either a zero of $f_1(\lambda)$ or $f_3(\lambda)$, but not of both.

Suppose λ_s is a zero of $f_1(\lambda)$. Then $f_2(\lambda_s) \neq 0$ and

$$f_2(\lambda_t)/f_2(\lambda_s) < 0. \tag{9}$$

Between λ_s and λ_t, $f_3(\lambda)$ does not vanish. Moreover, $f_3(\lambda_s) \neq 0$ and $f_3(\lambda_t) = 0$; consequently, Lemma 2 is contradicted unless

$$f_3(\lambda_s)/f_3'(\lambda_t) < 0 \ (> 0), \quad \text{when} \quad \lambda_s < \lambda_t \ (\lambda_s > \lambda_t). \tag{10}$$

On the other hand, if λ_s is a zero of $f_3(\lambda)$, then $f_4(\lambda_s) \neq 0$ and

$$f_4(\lambda_t)/f_4(\lambda_s) < 0. \tag{11}$$

Again utilizing Lemma 2, we have

$$f_1(\lambda_s)/f_1'(\lambda_t) < 0 \ (> 0), \quad \text{when} \quad \lambda_s < \lambda_t \ (\lambda_s > \lambda_t). \tag{12}$$

The additional inequalities

$$\text{(a)} \quad W_{12}(\lambda_s)W_{34}(\lambda_t) < 0, \qquad \text{(b)} \quad W_{12}(\lambda_t)W_{34}(\lambda_s) < 0 \tag{13}$$

are seen to follow from (H_1), once it is realized that (D_5) and (5) imply $W_{12}(\lambda)$ and $W_{34}(\lambda)$ are both of constant sign over $\Lambda_1 < \lambda < \Lambda_2$.

We now return to (8) and deduce that

$$P(\lambda_s, \lambda_t) = \begin{cases} W_{12}(\lambda_s)W_{34}(\lambda_t) \dfrac{f_2(\lambda_t)f_3(\lambda_s)}{f_2(\lambda_s)f_3'(\lambda_t)} & \text{when} \quad f_1(\lambda_s) = 0 \\ W_{12}(\lambda_t)W_{34}(\lambda_s) \dfrac{f_4(\lambda_t)f_1(\lambda_s)}{f_4(\lambda_s)f_1'(\lambda_t)} & \text{when} \quad f_3(\lambda_s) = 0 \end{cases} \tag{14}$$

From (9), (11), and (13) it is manifest that the coefficients of $f_3(\lambda_s)/f_3'(\lambda_t)$ and $f_1(\lambda_s)/f_1'(\lambda_t)$ in the right hand members here, are positive. Hence, recalling (10) and (12), we further see that

$$P(\lambda_s, \lambda_t) \quad \text{is} \quad \begin{cases} < 0 & \text{when} \quad \lambda_s < \lambda_t \\ > 0 & \text{when} \quad \lambda_s > \lambda_t \end{cases} \tag{15}$$

irrespective of which function, $f_1(\lambda)$ or $f_3(\lambda)$, has λ_s as a zero.

With this result in mind, it is evident that (7) implies

$$F(\lambda_s)F'(\lambda_t) \quad \text{is} \quad \begin{cases} > 0 & \text{when} \quad \lambda_s < \lambda_t \\ < 0 & \text{when} \quad \lambda_s > \lambda_t \end{cases} \tag{16}$$

It is now a straightforward matter to confirm that $F(\lambda)$ satisfies the hypotheses of Lemma 2 and, therefore, has at least one zero between λ_s and λ_t. Conclusion (c_3) is, thus, substantiated.

Case III. In this case, λ_s and λ_t are both double zeros of $f_1(\lambda)f_3(\lambda)$ and, therefore, are zeros of $F(\lambda)$ also.

Using (1), (4), (D_1), and (D_3) we derive the relation

$$w_{24}(\lambda_s)w_{24}(\lambda_t) = f_2(\lambda_s)f_2(\lambda_t)f_4(\lambda_s)f_4(\lambda_t)F'(\lambda_s)F'(\lambda_t) > 0. \tag{17}$$

If either λ_s or λ_t were a zero of $F(\lambda)$ of order $k < 1$, then (D_4) would be violated. Should either zero have order $k > 1$, (17) would be contradicted. Hence, λ_s and λ_t are simple zeros of $F(\lambda)$ and it follows that on (Λ_1, Λ_2):

(c_4) a double zero of $f_1(\lambda)f_3(\lambda)$, immediately preceded or succeeded by another double zero of the function, is a simple zero of $F(\lambda)$.

We shall also prove that:

(c_5) between any two consecutive double zeros of $f_1(\lambda)f_3(\lambda)$ on (Λ_1, Λ_2) there lies precisely one zero of $F(\lambda)$, and it is of order one.

An argument almost identical with the one used in establishing (c_3) of Case *II*, when applied to the open interval between λ_s and λ_t instead of to a half open interval, reveals that, in the present case, it is still true that between λ_s and λ_t :

$F(\lambda)$ can have only simple zeros, and
$F(\lambda)$ can have at most one zero.

We proceed by observing that the product

$$f_2(\lambda_s)f_2(\lambda_t)f_4(\lambda_s)f_4(\lambda_t) \tag{18}$$

is positive. Hence, in virtue of (17),

$$F'(\lambda_s)F'(\lambda_t) > 0. \tag{19}$$

With this result known, it becomes a trivial matter to verify that $F(\lambda)$ fulfills the hypotheses of Lemma 3; consequently, $F(\lambda)$ has at least one zero between λ_s and λ_t . The validity of (c_5) is now self-evident.

By way of summary, let us stress the fact that the five conclusions (c_1) through (c_5) suffice to establish (C_1) and (C_2). The proof of Theorem 3 is, thus, complete.

4. Applications of the theorems. Our theorems are especially suited to the problem of isolating the characteristic numbers of boundary value problems which, upon separation of variables, lead to Sturm-Liouville differential systems involving interface boundary conditions. For, by utilizing Laplace's generalized method for expanding determinants [2], it is possible to express the characteristic equations of such systems in the form of equation (1), [3, 4, 5, 6, 7, 8]. Indeed, when the system entails more than one pair of interface conditions, the functions $f_i(\lambda)$, appearing in (1), will themselves have the same structure as $F(\lambda)$. In this event, successive reapplications of Theorems 1 through 4 may be required in order to effect the ultimate isolation, and determination within intervals, of the characteristic numbers of the problem. As an aid in better clarifying ideas, let us consider an illustrative example.

Example. A composite shaft, of unit length and consisting of two adjoining segments, is undergoing free torsional vibrations. One segment of the shaft is tapered and it extends from a free end at $\chi = 0$ to the interface of the two segments at $\chi = \beta$, $0 < \beta < 1$. The cross section of this segment, at a distance χ from the free end, is a circle of radius $b\chi$. The other segment of the shaft is a right circular cylinder of radius r_0 and length $(1 - \beta)$. Its end at $\chi = 1$ is built-in. At the interface, the angle of twist, and the transmitted torque, are continuous functions.

It can be shown that the characteristic equation of such a shaft is given by (1) with

$$f_1(\lambda) = (2/\pi)^{1/2}\beta^{-3}\lambda^{-3/2}[\sin \beta\lambda - \beta\lambda \cos \beta\lambda], \tag{20}$$

$$f_2(\lambda) = -(2/\pi)^{1/2}\beta^{-4}\lambda^{-3/2}[(3 - \beta^2\lambda^2) \sin \beta\lambda - 3\beta\lambda \cos \beta\lambda], \tag{21}$$

$$f_3(\lambda) = \sin (1 - \beta)c\lambda, \tag{22}$$

$$f_4(\lambda) = -\mu c\lambda \cos (1 - \beta)c\lambda, \tag{23}$$

$$F(\lambda) \equiv f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda), \tag{24}$$

where c and μ are positive constants whose values depend upon those of the physical parameters of the problem [8]. Only positive values of λ will be allowed; for, otherwise, the values of $f_1(\lambda)$ and $f_2(\lambda)$ would not remain defined and real.

From the elementary nature of $f_3(\lambda)$ and $f_4(\lambda)$, it is at once evident that the zeros of these functions separate one another on every interval which does not include the origin. This same conclusion also follows from Theorem 2; because, when μ and c are positive and $0 < \beta < 1$, the right hand member of

$$W_{34}(\lambda) = -\frac{\mu c}{2}[2(1-\beta)c\lambda - \sin 2(1-\beta)c\lambda] \tag{25}$$

can vanish if, and only if, $\lambda = 0$.

In particular, when $\lambda > 0$, as we have required, $\sin 2(1-\beta)c\lambda < 2(1-\beta)c\lambda$ and, hence,

$$W_{34}(\lambda) < 0. \tag{26}$$

Now, return to (20) and (21) and compute $f_1'(\lambda)$ and $f_2'(\lambda)$. Then, substitute these expressions into (3), and utilize elementary trigonometric identities, to obtain

$$W_{12}(\lambda) = \frac{2}{\pi}\beta^{-5}\lambda^{-2}\left[(\beta^2\lambda^2 - 1) + \cos 2\beta\lambda + \frac{\beta\lambda}{2}\sin 2\beta\lambda\right], \tag{27}$$

and, thence,

$$W_{12}(\lambda) = \frac{2}{\pi}\beta^{-5}\lambda^{-2}\left[(\beta^2\lambda^2 - 1) + \left(1 + \frac{\beta^2\lambda^2}{4}\right)^{1/2}\sin\left(2\beta\lambda + \operatorname{Tan}^{-1}\frac{2}{\beta\lambda}\right)\right]. \tag{28}$$

We are interested in determining those positive values of λ which make $W_{12}(\lambda)$ positive and, hence, the product $W_{12}(\lambda)W_{34}(\lambda)$ negative. For, then, Theorems 1, 3, and 4 will be applicable. Thus, we require that the inequality

$$(\beta^2\lambda^2 - 1) + \left(1 + \frac{\beta^2\lambda^2}{4}\right)^{1/2}\sin\left(2\beta\lambda + \operatorname{Tan}^{-1}\frac{2}{\beta\lambda}\right) > 0$$

shall be satisfied. This condition is clearly fulfilled if $\beta\lambda > 1$ and

$$\beta^2\lambda^2 - 1 > \left(1 + \frac{\beta^2\lambda^2}{4}\right)^{1/2}. \tag{29}$$

From this relation it follows that if

$$3/2\beta < \lambda < \infty, \tag{30}$$

then

$$W_{12}(\lambda) > 0, \quad \text{and from (26)}, \quad W_{12}(\lambda)W_{34}(\lambda) < 0. \tag{31}$$

In view of Theorems 1, 3, and 4 we may, therefore, assert that on the interval (30):

(a_1) the zeros of $f_1(\lambda)$ and $f_2(\lambda)$ separate one another, and
(a_2) the zeros of $F(\lambda)$ interlace the zeros of $f_1(\lambda)f_3(\lambda)$, as well as the zeros of $f_2(\lambda)f_4(\lambda)$.

Note that neither μ nor c affect the determination of (30); however, the value of β does. In fact, it is obvious that $3/2\beta$ increases as β decreases. Consequently, (a_1) and (a_2) are of little practical value when β is a number very near to zero. This difficulty may be overcome, however, in the following way.

Return to (27) and replace cos $2\beta\lambda$ and sin $2\beta\lambda$ by their respective power series and, subsequently, collect coefficients of like powers of $\beta\lambda$ to obtain

$$W_{12}(\lambda) = \frac{1}{\pi}\beta^{-5}\lambda^{-2}\sum_{k=2}^{\infty}\frac{(-1)^k(k-1)}{(2k+2)!}(2\beta\lambda)^{2k+2}$$

This series is clearly alternating; hence, $W_{12}(\lambda)$ will surely be positive if

$$(2\beta\lambda)^6/6! - 2(2\beta\lambda)^8/8! > 0,$$

i.e., if

$$\lambda < 7^{1/2}/\beta. \tag{32}$$

Upon combining this result with that of (30), we see that (a_1) and (a_2) actually apply on every finite interval lying wholly along the positive half axis, $0 < \lambda < \infty$.

When $\beta = \frac{1}{4}$, $\mu = 1$, and $c = 1$, the zeros of the functions (20) through (24), for values of λ between 0 and 100, are listed in Table 1.

TABLE I.
Consecutive zeros on the interval $0 < \lambda < 100$ of the functions:

$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$f_4(\lambda)$	$F(\lambda)$
17.973637	.12000000	4.1887901	2.0943950	1.9630415
30.901008	23.053838	8.3775804	6.2831853	5.8775788
43.616488	36.380046	12.566370	10.471975	9.7504704
56.264776	49.291764	16.755161	14.660766	13.520898
68.883024	62.058416	20.943952	18.849556	17.036534
81.485212	74.756148	25.132741	23.038346	20.085278
94.077812	87.415500	29.321532	27.227137	23.042222
		33.510322	31.415926	26.341580
		37.699112	35.604717	29.694024
		41.887902	39.793508	32.802620
		46.076693	43.982297	35.803236
		50.265482	48.171088	39.014560
		54.454273	52.359878	42.301225
		58.643064	56.548668	45.426044
		62.831853	60.737458	48.458434
		67.020644	64.926249	51.639164
		71.209434	69.115038	54.890978
		75.398224	73.303829	58.022440
		79.587014	77.492620	61.075562
		83.775805	81.681409	64.242176
		87.964594	85.870200	67.472529
		92.153385	90.058990	70.607344
		96.342176	94.247780	73.674756
			98.436570	76.833713
				80.049557
				83.186300
				86.264088
				89.418433
				92.623824
				95.761774
				98.847418

By inspecting the entries of columns 1 and 2, and of columns 3 and 4, it is possible to detect the separation properties of Theorems 1 and 2. Similarly, the interlacing properties of Theorems 3 and 4 become apparent when a comparison is made of the entries in columns 1, 3, and 5, and in columns 2, 4, and 5. By plotting these corresponding sets of data, in turn, along directed line segments, each extending from $\lambda = 0$ to $\lambda = 100$, the separation and interlacing properties of these theorems may be depicted geometrically.

References

1. E. L. Ince, *Ordinary differential equations*, Dover, New York, 1926
2. C. C. MacDuffee, *The theory of matrices*, Chelsea, New York, 1956
3. W. M. Whyburn, *Differential systems with boundary conditions at more than two points*, Proc. of the Conference on Differential Equations, U. of Maryland, March 1955, pp. 1-21
4. T. J. Pignani and W. M. Whyburn, *Differential systems with interface and general boundary conditions*, J. of the Elisha Mitchell Scientific Society **72** (1956) 1
5. R. E. Smithson, O. N. Strand, and C. J. Thorne, *Temperatures in structures*, NAVORD Report 5558, Parts 2, 3, and 4, U. S. Naval Ordnance Test Station, China Lake, California, 1958
6. L. C. Barrett and C. J. Thorne, *Sturm-Liouville systems with generalized boundary conditions*, NAVORD Report 6517, U. S. Naval Ordnance Test Station, China Lake, California, 1959
7. L. C. Barrett, *Sturm-Liouville systems involving generalized boundary conditions*, Proc. South Dakota Acad. Sci. **38** (1959) 223
8. L. C. Barrett, *Separation and interlacing theorems with applications to the characteristic numbers of two-layer Sturm-Liouville systems*, NAVWEPS Report 8466, U. S. Naval Ordnance Test Station, China Lake, California, 1964