A CLASS OF REDUCIBLE SYSTEMS OF QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS*

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1. Introduction. Quasi-linear partial differential equations, occurring frequently in engineering problems, are often difficult to solve. This note presents a class of systems of quasi-linear equations reducible to a single linear heat equation, and gives an example of viscous fluid flow.

2. Reduction to a linear equation. The system of \( n \) equations under consideration is of the form

\[
\frac{\partial u_i}{\partial t} + F_i \frac{\partial u_i}{\partial x_i} = G_i \frac{\partial^2 u_i}{\partial x_i \partial x_i} + k \frac{\partial^2 u_i}{\partial x_i \partial x_i} + H_i R_i, \quad (i, j = 1, \ldots, n)
\]

where summation convention is adopted with index \( i \) not summed; \( F_i, G_i, \) and \( H_i \) are functions of \( u_i \) at least twice continuously differentiable; \( k \) is a constant and \( R_i \) a continuously differentiable function of \( t, x_1, \ldots, x_n \). With some restrictions on \( F_i, G_i, H_i, \) and \( R_i \), Eq. (1) can be reduced to a heat equation in \( n \) dimensions.

Consider the transformation

\[
F_i(u_i) = -\frac{2k}{\phi} \frac{\partial \phi}{\partial x_i}, \quad (2)
\]

corresponding to which the following relations are true:

\[
F_i \frac{\partial u_i}{\partial t} = \frac{2k}{\phi^2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x_i} - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial t \partial x_i},
\]

\[
F_i \frac{\partial u_i}{\partial x_i} = \frac{2k}{\phi^2} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial x_i \partial x_i},
\]

\[
F_i \frac{\partial^2 u_i}{\partial x_i \partial x_i} = \frac{4k}{\phi^3} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} + \frac{2k}{\phi^2} \frac{\partial^2 \phi}{\partial x_i \partial x_i} \frac{\partial \phi}{\partial x_i} + \frac{4k}{\phi^2} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial x_i \partial x_i} - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial x_i \partial x_i} - \frac{4k^2}{\phi^2} F_i' \left( \frac{1}{\phi} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right) \left( \frac{1}{\phi} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right),
\]

where prime denotes differentiation with respect to \( u_i \), and index \( i \) is not summed. Substitution of (3) into (1) yields, with \( i \) not summed,

\[
2k \left( \frac{\partial \phi}{\partial x_i} - \phi \frac{\partial}{\partial x_i} \right) \left( \frac{\partial \phi}{\partial t} - k \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right) = F_i' H_i R_i
\]

\[
+ \frac{4k^2}{\phi^2 F_i''} \left( G_i F_i' - k F_i'' \right) \left( \frac{1}{\phi} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right) \left( \frac{1}{\phi} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right).
\]

Noting that the left-hand side of (4) is the partial derivative of a function with respect to \( x_i \), we set

\[
G_i F_i' - k F_i'' = 0, \quad F_i' H_i = 1, \quad R_i = -\partial P/\partial x_i,
\]

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where \( i \) is not summed, and rewrite (4) as

\[
\frac{\partial}{\partial x_i} \left[ 2k \left( \frac{\partial \phi}{\partial t} - k \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right) - P \right] = 0,
\]

which can be integrated to give

\[
\frac{\partial \phi}{\partial t} = k \frac{\partial^2 \phi}{\partial x_i \partial x_i} + \left[ c(t) + \frac{P}{2k} \right] \phi,
\]

a linear heat equation with linear heat generation. We have shown that solutions of a system of \( n \) quasi-linear equations (1) can be obtained from the solutions of a linear equation (6). It is noted, however, that (6) will not yield all solutions of (1) because of the limitation imposed on \( F_t \) by the transformation (2).

Thus, the system of equations (1) can be reduced to a single linear equation whenever (5), which may be termed the "reducibility conditions," is satisfied. A necessary and sufficient condition for the first two of (5) is that \( F_i, G_i, \) and \( H_i \) are derived from a generating function \( f_i(u_i) \) by the formulae

\[
F_i = \int u^i f_i(u) \, du,
\]

\[
G_i = k \frac{d}{du} \left( \ln f_i \right) \, du_i,
\]

\[
H_i = f_i^{-1},
\]

where \( i \) is not summed. A necessary and sufficient condition for the last of (5) is that the Strokes tensor \( S \) for \( R_i \) vanishes identically

\[
S_{ii} = \frac{\partial R_i}{\partial x_i} - \frac{\partial R_i}{\partial x_i} = 0.
\]

3. Navier-Stokes equations. As an example, let us consider the Navier-Stokes equations for incompressible fluid flow

\[
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_i}, \quad i, j = 1, 2, 3,
\]

where \( u_i \) is the velocity component in the \( x_i \) direction, \( p \) the pressure, \( \rho \) the density, and \( \nu \) the kinematic viscosity, which is assumed constant. It is easily checked that the reducibility conditions (5) are satisfied; hence through the transformation

\[
u_i = -\frac{2\nu}{\rho} \frac{\partial \theta}{\partial x_i}.
\]

Eq. (9) reduces to a linear heat equation

\[
\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x_i \partial x_i} + \frac{p(t, x_1, x_2, x_3)}{2\rho \nu} \frac{\partial \theta}{\partial x_i}.
\]

It should be noted that the Navier-Stokes equation with no pressure gradient was reduced to the heat conduction equation by Cole [1]. The one-dimensional case without the pressure term was studied by Burgers [2] and Cole [1].
It is permissible to view (11) as describing a mathematical model of some viscous flow and to solve (11) as an initial value problem in infinite space with prescribed pressure \( p(t, x_1, x_2, x_3) \); then the velocity field so obtained will need a corresponding source distribution as given by

\[
Q(t, x_1, x_2, x_3) = -2\nu \frac{\partial^2 \ln \theta}{\partial x_i \partial x_j}
\]

to satisfy conservation of mass. On the other hand, when the source distribution is specified (this case being more physical), for instance \( Q = 0 \), Eq. (11) may be transformed into a Bernoulli's equation through elimination of \( \nabla^2 \theta \). This result is not surprising since in combination with the continuity equation the viscous term in (9) becomes \( \nu \nabla \times (\nabla \times u) \) that drops off under the assumption of irrotationality implied by (10). Conversely, the nonlinear Bernoulli's equation for inviscid flow

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \frac{\rho}{\rho} = 0
\]

may be converted into a linear heat equation similar to (11) by means of the equation of continuity and a change of variable \( \phi = \ln \theta \).

4. Some reducible equations. A few simple forms of (1) will be listed for reference. For simplicity of presentation, only one-dimensional equations are given. Corresponding to the generating functions \( f = 0, 1, e^u, nu^{-1}, \ln u, -\sin u, \) and \( \cos u \) in (7), the following equations belong to the reducible class (1):

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = k \frac{\partial^2 u}{\partial x^2} + R(t, x),
\]

\[
\frac{\partial u}{\partial t} + e^u \frac{\partial u}{\partial x} = k \left( \frac{\partial u}{\partial x} \right)^2 + k \frac{\partial^2 u}{\partial x^2} + e^{-u}R(t, x),
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{kn(n - 1)}{u} \left( \frac{\partial u}{\partial x} \right)^2 + k \frac{\partial^2 u}{\partial x^2} + \frac{R(t, x)}{nu^{n-1}},
\]

\[
\frac{\partial u}{\partial t} + u(\ln u - 1) \frac{\partial u}{\partial x} = \frac{k}{u \ln u} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial x^2} + \frac{R(t, x)}{\ln u},
\]

\[
\frac{\partial u}{\partial t} + \left( \cos u \right) \frac{\partial u}{\partial x} = -k \left( \cot u \right) \left( \frac{\partial u}{\partial x} \right)^2 + k \frac{\partial^2 u}{\partial x^2} + \left( -\csc u \right) R(t, x),
\]

where \( R(t, x) \) is any function integrable with respect to \( x \). It may be noted that in the one-dimensional case transformation (2) imposes no restriction on \( u \) more than the requirement for existence of solutions; hence every solution of the original equation may be obtained from the corresponding heat equation. Likewise, the \( n \)-dimensional systems may be derived from \( n \) generating functions, some or all of which may be identical. The one-dimensional equations listed above may provide a good visualization of the \( n \)-dimensional systems.
References
