SERIES-PARALLEL GROUNDED TWO-PORTS*

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1. Introduction. The synthesis of grounded two-ports without mutual inductance has been an outstanding unsolved problem of electric network theory for many years. The principal structure that has been utilized in the study of this problem has been the series-parallel structure. The present paper derives a number of properties of series-parallel two-ports. In particular a new realizability condition, independent of the residue and coefficient conditions, is obtained for *RC* series-parallel grounded two-ports. These results are described below.

As is well known, common factors play a decisive role in the synthesis of *RC* transfer functions when the reduced numerators of the transfer functions have some negative coefficients. Similarly, in two-port theory, a special position is occupied by those two-ports—called factorable networks—for which the unsimplified numerators and denominators of all the admittance functions and impedance functions have a common factor. These networks are discussed in §4. The main purpose of the paper is to study the effect on a given *RC* factorable network of two basic operations: (i) removal of an impedance which is series connected to the rest of the network at one of its external nodes (ii) decomposition of the network into two parallel networks. This is accomplished in §5 and §6.

Now every series-parallel network may be simplified at each stage of its decomposition by at least one of the above methods. Hence the results of §5 and §6 may be applied to these networks. This is done in §7. By this means the following theorem is proved (Theorem 7.1): *For an RC series-parallel factorable, grounded two-port, with common factor* $f(s)$, *the ratios* Δ_{12}/f, $(\Delta_{11} - \Delta_{12})/f$, $(\Delta_{22} - \Delta_{12})/f$ *are polynomials with non-negative coefficients.* Here Δ_{11}, Δ_{12}, Δ_{22} are the cofactors of the network admittance determinant. Some of the consequences of this new realizability criterion are explored. Thus, for example, we prove the result (Theorem 7.2): *Let the admittance functions* $Y_{11} = B^*/G^*$, $Y_{22}^* = C^*/G^*$, $-Y_{12}^* = D^*/G^*$ *of an RC series-parallel, grounded two-port be written so that* B^*, C^*, D^*, G^* *have no common factor and let* G_2 *be the factor of* G *which contains all non-compact poles of the network. Then* D^*G_2 *is a polynomial with non-negative coefficients.*

These results are independent of the residue and coefficient conditions and constitute further necessary properties of *RC* series-parallel networks. However, as indicated in §8, even all of these conditions do not constitute a set which is sufficient for series-parallel realizability. However if D^*, in Theorem 7.2 above, has non-negative coefficients, a method of series-parallel synthesis of Y_{11}, Y_{22}, KY_{12} is given for sufficiently small K. As is well known, the results for *RC* networks may be modified so as to apply to networks containing any two kinds of elements only.

2. Preliminaries. Let Γ be a general *RLC* transformerless network. The nodes of Γ

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are identified so that each branch consists of an R, L, and C in parallel. Consequently the admittance y_{ij} $(i \neq j)$ of the branch between nodes i and j is of the form $a + bs + c/s$ where s is the complex frequency variable, a, b and c are all non-negative and $y_{ij} = y_{ji}$. If Γ contains $t + 1$ nodes, also write

$$y_{ii} = \sum_{\substack{j=0 \\ j \neq i}}^{t} y_{ij} , \tag{2.1}$$

and introduce the notation

$$(ii) = y_{ii} , \qquad (ij) = -y_{ij} . \qquad (i \neq j) \qquad i, j = 0, 1, 2, \cdots , t$$

Let I_i be the current impressed by the driving sources upon node i and let E_i be the voltage from any fixed node (for example, node 0) taken as reference node to node i. The equations of the nodal system may be written as

$$I_i = \sum_{j=0}^{t} (ij)E_j , \qquad i = 0, 1, 2, \cdots , t$$

where, of course, $E_0 \equiv 0$. The network determinant $\mathfrak{D}$ is

$$\mathfrak{D} = |(ij)|, \qquad i, j = 0, 1, 2, \cdots , t$$

and equals zero, since (2.1) is equivalent to $\sum_{j=0}^{t} (ij) = 0$. Let $\mathfrak{D}_{ij}$ be the cofactor of (ij) in $\mathfrak{D}$, $\mathfrak{D}_{ijkl}$ be the cofactor of (kl) in $\mathfrak{D}_{ij}$, etc. Then, as shown in* [3, p. 58], all the $\mathfrak{D}_{ij}$ are equal. Also,

$$\mathfrak{D}_{iijj} = \mathfrak{D}_{iijk} + \mathfrak{D}_{jjik}. \qquad i, j, k = 0, 1, 2, \cdots t \tag{2.2}$$

Throughout the remainder of the paper, we suppose that the network Γ has only three external nodes which we take to be nodes 0, 1 and 2. To promote economy of language, we shall continue to refer to this three terminal network or grounded two-port simply as a network. In this case, we introduce the following additional notation:

$$A = \mathfrak{D}_{1122} , \qquad B = \mathfrak{D}_{0022} = \Delta_{22} , \qquad C = \mathfrak{D}_{0011} = \Delta_{11} ,$$
$$D = \mathfrak{D}_{0012} = \Delta_{12} , \qquad E = \mathfrak{D}_{1102} , \qquad F = \mathfrak{D}_{2201} \qquad G = \mathfrak{D}_{001122} = \Delta_{1122} . \tag{2.3}$$

It follows from (2.2) that

$$A = E + F, \qquad B = D + F, \qquad C = D + E \tag{2.4}$$

From (2.4) it is immediate that the three functions remaining from among A, B, C, D, E, F may be determined by any triplet of independent functions such as A, B, C or D, E, F or B, C, D or D, E, A. Furthermore, Jacobi's theorem applied to the cofactors of Δ yields $BC - D^2 = G\Delta$. After use of (2.4), this last equation may be written in the equivalent forms

$$\begin{aligned} BC - D^2 = CA - E^2 = AB - F^2 = G\Delta = DE + EF + FD \\ = \tfrac{1}{4}\{[A^2 + B^2 + C^2] - [(A - B)^2 + (B - C)^2 + (C - A)^2]\} \end{aligned} \tag{2.5}$$

The external behavior of Γ is usually characterized by the triplet of impedance functions

*The brackets refer to the references at the end of the paper.

$$Z_{11} = \frac{C}{\Delta}, \qquad Z_{22} = \frac{B}{\Delta}, \qquad Z_{12} = \frac{D}{\Delta} \tag{2.6}$$

or, equivalently, by the triplet of admittance functions

$$Y_{11} = \frac{B}{G}, \qquad Y_{22} = \frac{C}{G}, \qquad Y_{12} = -\frac{D}{G} \tag{2.7}$$

In both of these characterizations, node 0 occupies a special position. (The two-port Γ has node 0 as common to both the input and output terminals). However, the theory of the network with three external nodes may also be given a symmetric formulation by basing it on the quotients of A, B, C or alternatively, of D, E, F with either Δ or G.

3. Series-parallel networks. Every series-parallel network (and only these) is capable of further simplification at each stage either by parallel decomposition into two simpler networks of the same kind or by the series removal of an impedance at an external node leaving a simpler network of the same kind. We now consider the effect of these two operations upon the network determinant and its (repeated) cofactors.

First suppose that Γ (with external nodes 0, 1, 2) consists of the series connection of the admittance y joining node 0 to node 3 and the network Γ', whose external nodes are 1, 2, 3. Here node 3 in Γ' will also be written as node $0'$, since its role is analogous to that of node 0 in Γ. The network determinants of Γ and Γ' are

$$\mathfrak{D} = \begin{vmatrix} y & 0 & 0 & -y & \cdots & 0 & \cdots & 0 \\ 0 & (11) & (12) & (13) & & (14) & \cdots & (1t) \\ 0 & (21) & (22) & (23) & & (24) & \cdots & (2t) \\ -y & (31) & (32) & (33)+y & & (34) & \cdots & (3t) \\ 0 & (41) & (42) & (43) & & (44) & \cdots & (4t) \\ & & & & \cdots & & & \\ 0 & (t1) & (t2) & (t3) & & (t4) & \cdots & (tt) \end{vmatrix}, \tag{3.1}$$

and

$$\mathfrak{D}' = \begin{vmatrix} (33) & (31) & (32) & (34) & \cdots & (3t) \\ (13) & (11) & (12) & (14) & \cdots & (1t) \\ (23) & (21) & (22) & (24) & \cdots & (2t) \\ (43) & (41) & (42) & (44) & \cdots & (4t) \\ & & & \cdots & & \\ (t3) & (t1) & (t2) & (t4) & \cdots & (tt) \end{vmatrix}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\Delta = \begin{vmatrix} (11) & (12) & (13) & & (14) & \cdots & (1t) \\ (21) & (22) & (23) & & (24) & \cdots & (2t) \\ (31) & (32) & (33)+y & & (34) & \cdots & (3t) \\ (41) & (42) & (43) & & (44) & \cdots & (4t) \\ & & & \cdots & & & \\ (t1) & (t2) & (t3) & & (t4) & \cdots & (tt) \end{vmatrix} = \mathfrak{D}' + y\mathfrak{D}'_{33},$$

or, since $\mathfrak{D}' = 0$, $\mathfrak{D}'_{33} \equiv \mathfrak{D}'_{00} = \Delta'$, that

$$\Delta = y\Delta'. \tag{3.3}$$

Similarly,

$$\mathfrak{D}_{00\alpha\beta} = \mathfrak{D}'_{\alpha\beta} + y\mathfrak{D}'_{\alpha\beta 33}, \qquad \alpha, \beta = 1, 2, 4, 5, \cdots, t$$

or, since $\mathfrak{D}'_{\alpha\beta} = \Delta'$, $\mathfrak{D}'_{\alpha\beta 33} \equiv \mathfrak{D}'_{00\alpha\beta}$

$$\mathfrak{D}_{00\alpha\beta} = \Delta' + y\mathfrak{D}'_{00\alpha\beta}, \qquad \alpha, \beta = 1, 2, 4, 5, \cdots, t \tag{3.4}$$

Also,

$$\mathfrak{D}_{001122} = \mathfrak{D}'_{1122} + y\mathfrak{D}'_{112233},$$

or

$$\mathfrak{D}_{001122} = \mathfrak{D}'_{1122} + y\mathfrak{D}'_{001122}. \tag{3.5}$$

From (2.2) and (3.4), we find that

$$\begin{aligned} \mathfrak{D}_{\alpha\alpha 0\beta} &= y\mathfrak{D}'_{\alpha\alpha 0\beta} \\ \mathfrak{D}_{\alpha\alpha\beta\beta} &= y\mathfrak{D}'_{\alpha\alpha\beta\beta}, \qquad \alpha, \beta = 1, 2, 4, 5, \cdots, t \end{aligned} \tag{3.6}$$

In the notation of (2.3), equations (3.3), (3.4), (3.5), (3.6) become

$$\begin{aligned} &\Delta = y\Delta', \quad A = yA', \quad E = yE', \quad F = yF', \quad B = \Delta' + yB', \\ &C = \Delta' + yC', \quad D = \Delta' + yD', \quad G = A' + yG'. \end{aligned} \tag{3.7}$$

If the admittance y is removed at nodes 1 or 2, the corresponding results are obtained by interchanging subscripts in equations (3.3) to (3.6).

Now assume that Γ consists of the parallel connection of two networks Γ' and Γ''. If the network determinants of Γ' and Γ'' are written as

$$\begin{aligned} \mathfrak{D}' &= |(pq)'|, \qquad p, q = 0, 1, 2, 3, \cdots, a \\ \mathfrak{D}'' &= |(uv)''|, \qquad u, v = 0, 1, 2, a+1, a+2, \cdots, t \end{aligned}$$

then the corresponding determinant of Γ is

$$\mathfrak{D} = |(ij)|, \qquad i, j = 0, 1, 2, 3, \cdots, t$$

where the (ij) are given by

$$\begin{aligned} &(\alpha\beta) = (\alpha\beta)' + (\alpha\beta)'', \qquad (hl) = 0, \\ &(\alpha k) = (\alpha k)', \qquad (hk) = (hk)', \\ &(\alpha l) = (\alpha l)'', \qquad (lm) = (lm)''. \end{aligned} \tag{3.8}$$

Here $\alpha, \beta; h, k; l, m$ have the restricted ranges $0, 1, 2; 3, 4, \cdots, a; a+1, a+2, \cdots, t$ respectively, while i, j have the total range $0, 1, 2, \cdots, t$ and $(ij) = (ji)$. A Laplace expansion of the determinants defining B, C, D and G, after using (3.8), yields the results

$$\begin{aligned} &B = B'G'' + B''G', \qquad C = C'G'' + C''G', \\ &D = D'G'' + D''G', \qquad G = G'G'', \\ &\Delta = \Delta'G'' + \Delta''G' + B'C'' + B''C' - 2D'D'', \end{aligned} \tag{3.9}$$

where the primed quantities refer to the networks Γ' and Γ''. The last of these equations follows from the earlier ones after use is made of (2.5). Also, from (2.4) it follows that A, E, F have the same decomposition law as B, C, D.

4. Factorable networks. Examples of networks exist in which a common factor is present in only four or fewer of the network functions (2.3) without being a factor of any of the remaining functions. However, if all five of the functions B, C, D, G, Δ which occur in the impedance and admittance functions Z_{11}, Z_{12}, Z_{22}; Y_{11} Y_{12}, Y_{22} of Γ have a common factor, then it follows from (2.4) that all eight network functions must have this common factor. (Of course, this factor does not appear in the impedance or admittance functions). This same conclusion also holds if G, Δ and any three independent functions chosen from A, B, C, D, E, F have a common factor. We call a network Γ (with three external nodes) a *factorable network* if all eight network functions A, B, C, D, E, F, G, Δ have a common factor.

Examples of factorable networks may be constructed without difficulty as follows: Let Γ' be any network with external nodes 0, 1, 2 and let Γ^* be any network with an external node a. Let Γ', Γ^* have no common nodes.

(i) As a first example, modify Γ^* by adding the external node 0 connected to node a by an admittance y. Call the modified network Γ'' and let Γ be the network consisting of Γ' and Γ'' with common node 0. Then, as indicated in [3, p. 63], Δ of Γ may be partioned as

$$\Delta = \left|\begin{array}{c|c} \Delta' & 0 \\ \hline 0 & \Delta'' \end{array}\right|.$$

Hence

$$\Delta = \Delta'\Delta'', \qquad B = B'\Delta'', \qquad C = C'\Delta'', \qquad D = D'\Delta'', \qquad G = G'\Delta'',$$

where the unprimed quantities refer to Γ and the primed quantities to Γ', Γ''. Thus Γ is a factorable network. Of course, a similar result obtains if nodes 1 or 2 play the role of node 0 in the above discussion.

(ii) As a second example, modify Γ^* by adding external node 0 (unconnected to the other nodes of Γ^*) to form Γ''. Connect node a of Γ'' to any internal node b of Γ' by an admittance y to form the network Γ. Then the Δ of Γ may be partitioned as

$$\Delta = \left|\begin{array}{c|c|c|c} \Delta'_{bb} & \begin{array}{c} -y'_{1b} \\ -y'_{2b} \\ \cdots \end{array} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{\begin{array}{cc} -y'_{b1} - y'_{b2} \cdots y'_{bb} + y & \\ 0 \quad 0 \cdots -y & \end{array}} & \begin{array}{cc} -y & 0 \cdots 0 \\ y''_{aa} + y - y''_{a,a+1} & \cdots -y''_{at} \end{array} & \\ \hline \multicolumn{2}{c|}{0} & \begin{array}{c} -y''_{a+1,a} \\ \cdots \\ -y''_{ta} \end{array} & \Delta''_{aa} \end{array}\right|.$$

Now decompose column b into the sum of the columns $[-y'_{1b}, -y'_{2b} \cdots, y'_{bb}, 0, 0, \cdots 0]$ and $[0, 0, \cdots, y, -y, 0, \cdots, 0]$ and decompose column a similarly. Then Δ may be written as a sum of determinants

$$\Delta = \left|\begin{array}{c|c} \Delta' & \begin{array}{c|c} & \\ \hline -y & \\ \hline y & \end{array} \\ \hline 0 & \begin{array}{c} 0 \\ \hline -y''_{a,a+1} \cdots -y''_{at} \\ \hline \Delta''_{aa} \end{array} \end{array}\right| + \left|\begin{array}{c|c} \Delta' & 0 \\ \hline 0 & \Delta'' \end{array}\right|$$

$$+ \left|\begin{array}{c|c} \begin{array}{c} \Delta'_{bb} \\ \hline -y_{b1} - y_{b2} \cdots \end{array} & \begin{array}{c|c} & \\ \hline y & -y \\ \hline -y & y \end{array} \\ \hline 0 & \begin{array}{c} 0 \\ \hline -y''_{a,a+1} \cdots -y''_{at} \\ \hline \Delta''_{aa} \end{array} \end{array}\right| + \left|\begin{array}{c|c} \begin{array}{c} \Delta'_{bb} \\ \hline -y_{b1}' - y_{b2}' \cdots \end{array} & \begin{array}{c|c} 0 & \\ \hline y & \\ \hline -y & \end{array} \\ \hline 0 & \Delta'' \end{array}\right| .$$

This leads to

$$\Delta = y\Delta'\Delta''_{aa} + \Delta'\Delta'' + 0 + y\Delta'_{bb}\Delta''$$

Since node 0 in Γ'' is not connected to the rest of Γ'' it follows from [3, p. 60] that $\Delta'' = 0$. Hence the last equation becomes $\Delta = y\Delta'\Delta''_{aa}$. Similarly

$$B = yB'\Delta''_{aa}, \qquad C = yC'\Delta''_{aa}, \qquad D = yD'\Delta''_{aa}, \qquad G = yG'\Delta''_{aa}$$

so that Γ is a factorable network.

These two examples show that any network Γ' may be converted into a factorable network Γ by connecting any node of Γ' by an admittance to one node of another network Γ'' having no common nodes with Γ'. It is clear that the impedances Z_{11}, Z_{12}, Z_{22} and admittances Y_{11}, Y_{12}, Y_{22} of both Γ' and Γ are equal. However not every factorable network may be reduced to an equivalent network by the series removal at one of its nodes of an extraneous network. For example, the network Γ_0 whose Δ is given by

$$\Delta_0 = \begin{vmatrix} s+1 & 0 & -s & -1 \\ 0 & s+1 & -s & -1 \\ -s & -s & 2s+2 & 0 \\ -1 & -1 & 0 & 2s+2 \end{vmatrix},$$

is factorable (with common factor $s + 1$) but cannot be reduced in this manner to a non-factorable network. It is shown in [3, p. 65] that this network is not equivalent to any *RC* non-factorable network.

It is known that deletion of any factors from any of the functions A, B, C, G, Δ of an *RC* network always leaves a polynomial with positive coefficients. However examples have been given in which negative coefficients appear in D/f where f is a common factor of D and three other functions. Networks illustrating this property are given below, of which example 3 has not appeared before.

Example 1. [6, p. 92] B, C, D, Δ have common factor $s + 1$. $D/(s + 1) = 30s^2 - 15s + 30$.

$$\Delta = \begin{vmatrix} s+1 & 0 & -1 & -s \\ 0 & 3s+3 & -3 & -3s \\ -1 & -3 & 10s+5 & 0 \\ -s & -3s & 0 & 5s+10 \end{vmatrix}.$$

Example 2. [3, p. 64] C, D, G, Δ have common factor $s + 1$. $D/(s + 1) = 4s^2 - s + 4$.

$$\Delta = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & s+1 & 0 & -1 & -s \\ -1 & 0 & 2(s+1) & -1 & -s \\ 0 & -1 & -1 & 4s+3 & 0 \\ 0 & -s & -s & 0 & 3s+4 \end{vmatrix}.$$

Example 3. B, C, D, G have common factor $s + 1$. $B = C = (s + 1)(32s^3 + 116s^2 + 85s + 12)$, $D = (s + 1)(4s^2 - s + 4)$ $G = (s + 1)^2(32s^2 + 104s + 32)$.

$$\Delta = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2(s+1) & 0 & -1 & -s \\ 0 & -1 & 0 & 2(s+1) & -1 & -s \\ 0 & 0 & -1 & -1 & 4s+3 & 0 \\ 0 & 0 & -s & -s & 0 & 3s+4 \end{vmatrix}.$$

These three examples include all essentially different combinations of four network functions (including one transfer admittance function) to which a common factor may be assigned. This suggests the question whether negative coefficients can occur when f is a common factor of more than four network functions. Otherwise, does a factorable RC network exist with common factor f such that D/f, E/f or F/f has negative coefficients? We investigate aspects of this problem in the sequel.

5. *RC* series connected factorable networks. We now consider an RC network Γ which consists of an admittance y at node 0 in series with a network Γ'. Since the RC impedance $1/y$ may be considered as a series connection of simpler elements, each element being an R and C in parallel, we may assume without loss of generality that our y corresponds to the first of these elements, incorporating the others into Γ', and write $y = a + bs$. Also assume that Γ is factorable, so that the network functions A, B, C, D, E, F, G, Δ of Γ have a (non-constant) greatest common factor

$$f(s) = \prod_i x_i^{p_i}$$

where the x_i are the distinct factors of $f(s)$. The relation between the network functions of Γ and Γ' is given by (3.7). Concerning these networks we now prove the theorem:

THEOREM 5.1. *Let RC network Γ, consisting of admittance $y = a + bs$ in series with Γ' be factorable with common factor $f(s)$. Then the reduced network Γ' is factorable with common factor $f'(s) = f(s)/g$, where g is the greatest common divisor of $f(s)$ and y.*

We consider two cases for each distinct factor x_i :

(i) x_i does not divide y. Then it follows from the first line of (3.7) that A', E', F', Δ' are divisible by $x_i^{p_i}$ and then from the remaining equations of (3.7) that B', C', D', G' are also divisible by $x_i^{p_i}$.

(ii) x_i divides y. Then, since $y = kx_i$, where k is a constant, it follows from (3.7) that A', E, F', Δ' are all divisible by $x_i^{q_i}$, where $q_i = p_i - 1$. Also, from (3.7),

$$\frac{B}{\Delta} = \frac{1}{y} + \frac{B'}{\Delta'}.$$

Now B/Δ, $1/y$ and B'/Δ' are all RC impedances. Also the residue of $1/y$ at $x_i = 0$ is positive and the corresponding residue of B'/Δ' cannot be negative. Hence B/Δ has a pole at $x_i = 0$. Since B is divisible by $x_i^{p_i}$, Δ must be divisible by $x_i^{p_i+1}$. Then, from (3.7), Δ' is divisible by $x_i^{q_i}$. It follows from (3.7) that B', C', D' are all divisible by $x_i^{q_i}$. Finally, since

$$\frac{G}{A} = \frac{1}{y} + \frac{G'}{A'}.$$

we find in a similar manner that G/A has a pole at $x_i = 0$, so that A must be divisible by $x_i^{p_i+1}$. Then, from (3.7), we must have that G' is divisible by $x_i^{q_i}$. Consequently all the network functions of Γ' have the common factor $x_i^{q_i}$ in this case.

The results obtained in cases (i) and (ii) also hold if the role of node 0 is played by either node 1 or node 2 since only the names of the nodes are involved. This completes the proof of Theorem 5.1.

The preceding discussion also shows that, when y is removed at node 0, Δ' and A' are each divisible by $f(s)$. Consequently, we find from (3.7) that

$$\frac{E}{f} = \frac{y}{g}\cdot\frac{E}{f'}, \qquad \frac{F}{f} = \frac{y}{g}\cdot\frac{F'}{f'}$$

$$\frac{D}{f} = \frac{\Delta'}{f} + \frac{y}{g}\cdot\frac{D'}{f'}.$$

Now y/g and Δ'/f are polynomials in s with positive coefficients. (The same is true for A'/f, B'/f', C'/f', G'/f,). Then the preceding equations show that D/f, E/f or F/f can have a negative coefficient only if the same is true of the corresponding quotient D'/f', E'/f' or F'/f' respectively. We state this result in

THEOREM 5.2. *The presence of a negative coefficient in any of the quotients D/f, E/f or F/f of an RC series connected factorable network Γ implies the presence of a negative coefficient in the corresponding quotient D'/f', E'/f' or F'/f' respectively of the reduced network Γ'.*

6. *RC* parallel connected factorable networks. We now consider an RC network Γ which is the parallel connection of two networks Γ' and Γ''. Also assume that Γ is factorable so that its network functions A, B, C, D, E, F, G, Δ have a (non constant) greatest common factor

$$f(s) = \prod_i x_i^{\gamma_i}, \qquad \gamma_i > 0 \tag{6.1}$$

where the x_i are the distinct factors of $f(s)$. The dependence of the network functions of Γ on those of Γ' and Γ'' (which we indicate by one and two primes respectively) is given by (3.9). In particular, since $G = G'G''$, $f(s)$ must divide $G'G''$ so that we may write

$$G' = G_1\cdot\prod x_i^{\alpha_i}, \qquad G'' = G_2\cdot\prod x_i^{\beta_i}, \tag{6.2}$$

where G_1 and G_2 are each relatively prime to $f(s)$ and

$$\alpha_i + \beta_i \geqq \gamma_i\,, \qquad \alpha_i \geqq 0, \qquad \beta_i \geqq 0 \tag{6.3}$$

Now define the functions

$$\phi(s) = \prod x_i^{p_i}, \qquad \psi(s) = \prod x_i^{q_i}, \qquad \zeta(s) = \prod x_i^{r_i}, \tag{6.4}$$

where the definition of the non-negative exponents p_i, q_i, r_i for each value of i depends upon which of the following mutually exclusive, exhaustive inequalities holds for that value of i:

$$\alpha_i \geqq \gamma_i\,, \qquad \beta_i \geqq \gamma_i\,, \tag{6.5$_1$}$$

$$\alpha_i \geqq \gamma_i > \beta_i\,, \tag{6.5$_2$}$$

$$\beta_i \geqq \gamma_i > \alpha_i\,, \tag{6.5$_3$}$$

$$\gamma_i > \alpha_i\,, \qquad \gamma_i > \beta_i\,. \tag{6.5$_4$}$$

In these four cases, the corresponding values of p_i, q_i, r_i are:

$$p_i = q_i = 0, \qquad r_i = \gamma_i\,, \tag{6.6$_1$}$$

$$p_i = \gamma_i - \beta_i\,, \qquad q_i = 0, \qquad r_i = \beta_i\,, \tag{6.6$_2$}$$

$$p_i = 0, \qquad q_i = \gamma_i - \alpha_i\,, \qquad r_i = \alpha_i\,, \tag{6.6$_3$}$$

$$p_i = \gamma_i - \beta_i\,, \qquad q_i = \gamma_i - \alpha_i\,, \qquad r_i = \alpha_i + \beta_i - \gamma_i\,, \tag{6.6$_4$}$$

respectively. It is clear from (6.1), (6.4) and (6.6) that

$$f = \phi\psi\zeta. \tag{6.7}$$

Also, as follows from (6.2), (6.4) ,(6.5) and (6.6), G' and G'' are divisible by $\zeta\phi$ and $\zeta\psi$ respectively, so that we may write

$$G' = \zeta\phi G'^*, \qquad G'' = \zeta\psi G''^*. \tag{6.8}$$

In the sequel, we shall have occasion to develop proofs which subdivide according to the different inequalities (6.5). When this occurs, $(6.5)_3$ will be omitted from the discussion since any proof for case $(5.6)_2$ is valid for $(6.5)_3$ after interchange of the roles of α_i and β_i.

We shall prove the following theorem concerning the parallel connected network Γ:

THEOREM 6.1. *Let RC network Γ consisting of the parallel connection of networks Γ' and Γ'' be factorable with common factor $f(s)$. Then the component networks Γ', Γ'' are also factorable with common factors $\phi(s)$, $\psi(s)$ respectively, where* $f = \phi\psi\zeta$.*

We first prove that the network functions B', C', D'; B'', C'', D'' have the common factors $\phi(s)$, $\psi(s)$ respectively; that is

$$\begin{aligned} B' &= \phi B'^*, \qquad C' = \phi C'^*, \qquad D' = \phi D'^*, \\ B'' &= \psi B''^*, \qquad C'' = \psi C''^*, \qquad D'' = \psi D''^*. \end{aligned} \tag{6.9}$$

Since all the factors x_i are relatively prime, it will suffice to show that the functions B',

*The factor functions ϕ, ψ, ζ are defined by (6.4), (6.5) and (6.6).

C', D' and B'', C'', D'' are divisible by the factors $x_i^{p_i}$ and $x_i^{q_i}$ respectively for each value of i; that is,

$$\begin{aligned} B' &= x_i^{p_i+\sigma_1}B_i', & B'' &= x_i^{q_i+\tau_1}B_i'', \\ C' &= x_i^{p_i+\sigma_2}C_i', & C'' &= x_i^{q_i+\tau_2}C_i'', \\ D' &= x_i^{p_i+\sigma_3}D_i', & D'' &= x_i^{q_i+\tau_3}D_i'', \end{aligned} \tag{6.10}$$

where B_i', C_i', D_i', B_i'', C_i'', D_i'' are all relatively prime to x_i and

$$\sigma_j \geqq 0, \qquad \tau_j \geqq 0. \qquad j = 1, 2, 3 \tag{6.11}$$

This is immediate for factors arising under $(6.5)_1$. To discuss the remaining cases, we begin with $B = B'G'' + B''G'$ from (3.9) which, in virtue of (6.1) and (6.2), we may write as

$$x_i^{\gamma_i}B_i = x_i^{\beta_i}B'G_i'' + x^{\alpha_i}B''G_i' \tag{6.12}$$

where

$$B = x_i^{\gamma_i}B_i, \qquad G' = x^{\alpha_i}G_i', \qquad G'' = x_i^{\beta_i}G_i'' \tag{6.13}$$

and G_i', G_i'' are relatively prime to x_i.

For a factor satisfying $(6.5)_2$, division of both members of (6.12) by $x_i^{\beta_i}$ shows that $B'G_i''$ is divisible by $x_i^{\gamma_i-\beta_i}$. Since G_i'' is prime to x_i, it follows from $(6.6)_2$ that B', B'' obey (6.10), (6.11) in this case. If we begin with the equation of (3.9) for C or D instead of the one for B, we find similarly that C', D' and C'', D'' satisfy (6.10), (6.11) in case $(6.5)_2$.

To fix our ideas in case $(6.5)_4$, we assume without loss of generality that $\alpha_i \geqq \beta_i$. Then division of (6.12) by $x_i^{\beta_i}$ leads to the conclusion

$$B' = x_i^{\alpha_i-\beta_i+\delta_1}B_i', \qquad B'' = x_i^{\epsilon_1}B_i'', \tag{6.14}$$

where $\delta_1 \geqq 0$, $\epsilon_1 \geqq 0$ and B_i', B_i'' are prime to x_i. Similarly,

$$\begin{aligned} C' &= x_i^{\alpha_i-\beta_i+\delta_2}C_i', & C'' &= x_i^{\epsilon_2}\, C_i'', \\ D' &= x_i^{\alpha_i-\beta_i+\delta_3}D_i', & D'' &= x_i^{\epsilon_3}D_i'', \end{aligned} \tag{6.15}$$

where δ_2, δ_3, ϵ_2, ϵ_3 are also non-negative and C_i', C_i'', D_i', D_i'' are all prime to x_i. Now the ratios

$$\begin{aligned} \frac{B'}{G'} &= x_i^{\delta_1-\beta_i}\frac{B_i'}{G_i'}, & \frac{C'}{G'} &= x_i^{\delta_2-\beta_i}\frac{C_i'}{G_i'}, \\ \frac{B''}{G''} &= x_i^{\epsilon_1-\beta_i}\frac{B_i''}{G_i''}, & \frac{C''}{G''} &= x_i^{\epsilon_2-\beta_i}\frac{C_i''}{G_i''}, \end{aligned} \tag{6.16}$$

are all RC driving point admittances and the ratios

$$\frac{D'}{G'} = x_i^{\delta_3-\beta_i}\frac{D_i'}{G_i'}, \qquad \frac{D''}{G''} = x_i^{\epsilon_3-\beta_i}\frac{D_i''}{G_i''}, \tag{6.17}$$

are RC transfer admittances. It is known [1, 4] that the poles of such functions must be simple and, for driving point admittances, the zeros must also be simple. Consequently

$$|\delta_1 - \beta_i| \leqq 1, \qquad |\delta_2 - \beta_i| \leqq 1, \qquad |\epsilon_1 - \beta_i| \leqq 1, \qquad |\epsilon_2 - \beta_i| \leqq 1,$$
$$\delta_3 \geqq \beta_i - 1, \qquad \epsilon_3 \geqq \beta_i - 1.$$

Since these inequalities imply $\delta_j \geqq \beta_i - 1$, $\epsilon_j \geqq \beta_i - 1$ where $j = 1, 2, 3$, it follows from (6.3) that

$$\delta_j \geqq \gamma_i - \alpha_i, \qquad \epsilon_j \geqq \gamma_i - \alpha_i, \qquad j = 1, 2, 3 \tag{6.18}$$

unless $\gamma_i - \alpha_i = \beta_i$. Suppose that $\gamma_i - \alpha_i = \beta_i$. Now, from (6.13), the RC admittance

$$\frac{B}{G} = x_i^{\gamma_i - \alpha_i - \beta_i} \frac{B_i}{G_i' G_i''} = \frac{B_i}{G_i' G_i''},$$

has no pole at x_i. Also, from (3.9),

$$\frac{B}{G} = \frac{B'}{G'} + \frac{B''}{G''}$$

so that B'/G' and B''/G'' each cannot have a pole at x_i since the residue at a pole must be positive. It follows from (6.16) that $\delta_1 \geqq \beta_i = \gamma_i - \alpha_i$, $\epsilon_1 \geqq \beta_i = \gamma_i - \alpha_i$. Similarly, C/G has no pole at x_i and, from (6.16), $\delta_2 \geqq \gamma_i - \alpha_i$, $\epsilon_2 \geqq \gamma_i - \alpha_i$. Finally, it follows from the residue condition [1, 4] that D/G has no pole at x_i and, from (6.17) that $\delta_3 \geqq \gamma_i - \alpha_i$, $\epsilon_3 \geqq \gamma_i - \alpha_i$. This means that (6.18) is true without restrictions. Then equations (6.14) and (6.18) are equivalent to (6.10), (6.11), after account is taken of $(6.6)_4$. This completes the proof of (6.9) or, alternatively of (6.10), (6.11) for all possible cases. Of course, as follows from (2.4) and (6.9), A', E', F'; A'', E'', F'' also have the common factors $\phi(s)$, $\psi(s)$ respectively.

In the sequel, we prove that Δ', Δ'' are divisible by $\phi(s)$, $\psi(s)$ respectively; that is

$$\Delta' = \phi\Delta'^*, \qquad \Delta'' = \psi\Delta''^*. \tag{6.19}$$

If we write

$$\Delta' = x_i^{u_i}\Delta_i', \qquad \Delta'' = x_i^{v_i}\Delta_i'' \tag{6.20}$$

where $u_i \geqq 0$, $v_i \geqq 0$ and Δ_i', Δ_i'' are each relatively prime to x_i, then (6.19) is equivalent to

$$u_i \geqq p_i, \qquad v_i \geqq q_i, \tag{6.21}$$

for all values of i. Clearly (6.21) is true for those values of i where $(6.5)_1$ is satisfied, since $(6.6)_1$ holds.

Also, from (6.10), (6.13), (6.20)

$$\frac{B'}{G'} = x_i^{p_i + \sigma_1 - \alpha_i} \frac{B_i'}{G_i'} \qquad \frac{B''}{G''} = x_i^{q_i + \tau_1 - \beta_i} \frac{B_i''}{G_i''}$$
$$\frac{B'}{\Delta'} = x_i^{p_i + \sigma_1 - u_i} \frac{B_i'}{\Delta_i'} \qquad \frac{B''}{\Delta''} = x_i^{q_i + \tau_1 - v_i} \frac{B_i''}{\Delta_i''}$$

and analogous equations obtain with C', D'; C'', D'' replacing B', B'' respectively. Now these ratios are all RC admittances or RC impedances and consequently have simple poles and, for driving point functions, simple zeros. Consequently

$$|\alpha_i - p_i - \sigma_j| \leqq 1, \qquad |\beta_i - q_i - \tau_j| \leqq 1, \qquad j = 1, 2 \tag{6.22}$$
$$\sigma_3 \geqq \alpha_i - p_i - 1, \qquad \tau_3 \geqq \beta_i - q_i - 1,$$

and

$$u_i \geqq p_i + \sigma_j - 1, \qquad v_i \geqq q_i + \tau_j - 1. \qquad j = 1, 2 \tag{6.23}$$

We restrict the subsequent discussion to the case

$$\sigma_j = 0, \qquad \tau_j = 0, \qquad j = 1, 2 \tag{6.24}$$

since otherwise (6.23) would establish (6.21) and attain our objective. From (6.20), (6.22), (6.23) and (6.24),

$$\Delta' = x_i^{p_i - 1 + w_1} \Delta_i', \qquad \Delta'' = x_i^{q_i - 1 + w_2} \Delta_i'', \qquad w_1 \geqq 0, \qquad w_2 \geqq 0 \tag{6.25}$$

$$|\alpha_i - p_i| \leqq 1, \qquad |\beta_i - q_i| \leqq 1, \tag{6.26}$$

are true for those factors x_i still to be considered.

If $(6.5)_2$ is true for some i, then from (6.3) and $(6.6)_2$, we find that (6.26) becomes

$$\alpha_i + \beta_i - \gamma_i = 0 \quad \text{or} \quad 1, \qquad \beta_i = 0 \quad \text{or} \quad 1. \tag{6.27}$$

If, $\beta_i = 0$, then the first equation (6.27) becomes $\alpha_i = \gamma_i$ since $(6.5)_2$ holds. Then the last equation of (3.9), after simplification by (6.10) and (6.13), shows that Δ' is divisible by $x_i^{p_i}$, proving (6.21) for this subcase. Consequently we may restrict our consideration of (6.27) to $\beta_i = 1$. After using $(6.5)_2$, the remaining condition (6.27) becomes

$$\beta_i = 1, \qquad \gamma_i = \alpha_i, \tag{6.28}$$

for case $(6.5)_2$.

If $(6.5)_4$ is true for some i, then (6.3) and $(6.6)_4$ transform (6.26) into the equivalent condition $\alpha_i + \beta_i - \gamma_i = 0$ or 1. However, if $\alpha_i + \beta_i - \gamma_i = 0$, then the equations

$$B'C' - D'^2 = G'\Delta', \qquad B''C'' - D''^2 = G''\Delta'',$$

which are analogous to (2.5), readily yield (6.21) after substitution from $(6.6)_4$, (6.10), (6.13), (6.20) and (6.24). Consequently we restrict our discussion of case $(6.5)_4$ to the remaining condition

$$\alpha_i + \beta_i - \gamma_i = 1. \tag{6.29}$$

Returning to case $(6.5)_2$, subject to (6.28), we find that the last equation of (3.9) becomes

$$x_i\Delta^+ = x_i^{w_1}\Delta_i'G_i'' + x_i^{1+w_2}\Delta_i''G_i' + (B_i'C_i'' + B_i''C_i' - 2D'^+D''^+) \tag{6.30}$$

after using

$$\Delta = x_i^{\gamma_i}\Delta^+, \qquad D' = x_i^{p_i}D'^+, \qquad D'' = x_i^{q_i}D''^+, \tag{6.31}$$

and $(6.6)_2$, (6.10), (6.13), (6.24) and (6.25) and then dividing by $x_i^{\gamma_i - 1}$. Similarly, for case $(6.5)_4$, after use of $(6.6)_4$, (6.10), (6.13), (6.24), (6.25) and (6.29), the last equation of (3.9) becomes

$$x_i\Delta^+ = x_i^{w_1}\Delta_i'G_i'' + x_i^{w_2}\Delta_i''G_i' + (B_i'C_i'' + B_i''C_i' - 2D'^+D''^+). \tag{6.32}$$

Now, for case $(6.5)_2$, using $(6.6)_2$, (6.10), (6.13), (6.24) and (6.28)

$$\frac{B'}{G'} = \frac{B_i'}{x_iG_i'} = \frac{g'(s)}{G_i'} - \frac{b'}{x_i}, \tag{6.33}$$

where the last expression is the partial fraction expansion and $b' > 0$ is the residue of the admittance B'/G' at the pole $x_i = 0$. Equation (6.33) is also found to be true for case $(6.5)_4$, after use of $(6.6)_4$, (6.10), (6.13), (6.24) and (6.29). Also, in the same way for both cases, we find

$$\frac{B''}{G''} = \frac{B_i''}{x_i G_i''} = \frac{g''(s)}{G_i''} - \frac{b''}{x_i}, \tag{6.34}$$

where $b'' > 0$ is the residue of the admittance B''/G'' at the pole $x_i = 0$. Equations similar to (6.33), (6.34) also obtain for C'/G', D'/G'; C''/G'', D''/G'' respectively involving the residues c', d'; c'', d'' respectively.

For case $(6.5)_2$,

$$B'C' - D'^2 = G'\Delta' = x_i^{\alpha_i} G_i' x_i^{\gamma_i - 2 + w_1} \Delta_i' = x_i^{2\gamma_i - 2 + w_1} G_i' \Delta_i'$$

Division by G'^2 yields, after use of equations like (6.33),

$$\frac{x_i^{w_1}\Delta_i'}{x_i^2 G_i'} = \frac{B'C' - D'^2}{G'^2} = \frac{h(s)}{x_i G_i'} + \frac{(b'c' - d'^2)}{x_i^2},$$

and finally

$$\frac{x_i^{w_1}\Delta_i'}{x_i} = h(s) + \frac{(b'c' - d'^2)G_i'}{x_i} = k(s) + \frac{(b'c' - d'^2)(G_i')_0}{x_i}, \tag{6.35}$$

where $h(s)$, $k(s)$ are polynomials and $(G_i')_0$ is the value of G_i' for $x_i = 0$. In a similar manner

$$\frac{B'C'' + B''C' - 2D'D''}{G'G''} = \frac{x_i^{\gamma_i - 1}(B_i'C_i'' + B_i''C_i' - 2D'^+D''^+}{x_i^{\alpha+1} G_i' G_i''}$$

$$= \frac{B_i'C_i'' + B_i''C_i' - 2D'^+D''^+}{x_i^2 G_i' G_i''}.$$

Use of equations like (6.33) and (6.34) in the above equation yields

$$\frac{B_i'C_i'' + B_i''C_i' - 2D'^+D''^+}{x_i} = l(s) + \frac{(b'c'' + b''c' - 2d'd'')(G_i')_0(G_i'')_0}{x_i} \tag{6.36}$$

where $l(s)$ is a polynomial and $(G_i'')_0$ is the value of G_i'' for $x_i = 0$. For case $(6.5)_4$, parallel calculations to the above (but involving different details) establish the validity of the same equations (6.35) and (6.36) as well as of

$$\frac{x_i^{w_2}\Delta_i''}{x_i} = m(s) + \frac{(b''c'' - d''^2)(G_i'')_0}{x_i} \tag{6.37}$$

where $m(s)$ is a polynomial.

For case $(6.5)_2$, we substitute (6.35) and (6.36) in (6.30). This yields

$$\Delta^+ - x_i^{w_2}\Delta_i'' G_i' = H(s) + [(b'c' - d'^2) + (b'c'' + b''c' - 2d'd'')](G_i')_0(G_i'')_0/x_i$$

where $H(s)$ is a polynomial. Since $(G_i')_0(G_i'')_0 \neq 0$, we must have

$$(b'c' - d'^2) + (b'c'' + b''c' - 2d'd'') = 0. \tag{6.38}$$

Now $(b'c'' - b''c')^2 \geqq 0$ implies $b'^2c''^2 + 2b'b''c'c'' + b''^2c'^2 \geqq 4b'b''c'c''$. Since b', b'',

c', c'' are all non-negative, this last inequality implies $b'c'' + b''c' \geqq 2\sqrt{b'b''c'c''} \geqq 2d'd''$. The final inequality follows from the residue conditions for RC admittance functions, $b'c' - d'^2 \geqq 0$, $b''c'' - d''^2 \geqq 0$. Consequently (6.38) implies that

$$b'c' - d'^2 = 0, \qquad b'c'' + b''c' - 2d'd'' = 0 \tag{6.39}$$

Substitution of (6.39) in (6.35) shows that $w_1 \geqq 1$, so that from (6.25), the truth of (6.21) is established for case $(6.5)_2$.

For case $(6.5)_4$, we substitute (6.35), (6.36) and (6.37) in (6.32). Consequently

$$\Delta^+ = K(s) + [(b'c' - d'^2) + (b''c'' - d''^2) + (b'c'' + b''c' - 2d'd'')](G_i')_0(G_i'')_0/x_i ,$$

where $K(s)$ is a polynomial. Therefore

$$(b'c' - d'^2) + (b''c'' - d''^2) + (b'c'' + b''c' - 2d'd'') = 0. \tag{6.40}$$

The discussion following (6.38) shows that (6.40) implies (6.39) as well as

$$b''c'' - d''^2 = 0 \tag{6.41}$$

Substitution of (6.39) and (6.41) in (6.35) and (6.37) indicates that $w_1 \geqq 1$, $w_2 \geqq 1$. Then, from (6.25), the validity of (6.21) is established for case $(6.5)_4$. This completes the proof of (6.19). The conclusions (6.8), (6.9) and (6.19), which have been proved above, establish the validity of Theorem 6.1.

According to (3.9),

$$D = D'G'' + D''G'.$$

If we write $D = f(s)D^*$ and use (6.7), (6.8) and (6.9), the preceding equation becomes

$$D^* = D'^*G''^* + D''^*G'^*$$

It follows from this equation that D/f can have a negative coefficient only if the same is true of at least one of the quotients D'/ϕ or D''/ψ. Exactly similar conclusions hold for E/f and F/f. We state this result in the theorem:

THEOREM 6.2. *The presence of a negative coefficient in any of the quotients D/f, E/f, F/f of an RC parallel connected factorable network Γ implies the presence of a negative coefficient in the corresponding quotient D'/ϕ, E'/ϕ, F'/ϕ respectively of the component network Γ' or in the corresponding quotient D''/ψ, E''/ψ, F''/ψ respectively of the component network Γ''.*

7. *RC* series-parallel networks. When a series-parallel network is simplified by series removal of an admittance at an external node, the remaining network is reduced by one internal node. When the network is decomposed into two parallel networks, either one of them is a Π network or the number of internal nodes in the decomposed networks is less than in the composite network. Since at least one of these methods of simplification is possible at each stage, the reduction in the number of internal nodes of a finite series-parallel network means that eventually a stage will be reached where the reduced networks are all simple T-networks or Π-networks. A simple calculation of D', E', F' for these networks shows that there is no possibility for the appearance of any negative coefficients in D'/f, E'/f, or F'/f. These remarks in conjunction with Theorem 5.2 and Theorem 6.2 lead directly to the following new realizability criterion for RC series-parallel networks.

THEOREM 7.1. *For an RC series-parallel factorable network with common factor $f(s)$, the ratios D/f, E/f and F/f are polynomials with non-negative coefficients.*

It is possible that some coefficients in these ratios may be zero. Thus the network Γ_0 for which Δ_0 is given in §4 is an *RC* series-parallel factorable network with $f(s) = s + 1$ for which $D/f = 2s^2 + 2$. We now explore some consequences of Theorem 7.1. We first prove the following two theorems:

THEOREM 7.2. *Let Γ be an RC series-parallel network with admittance functions B^*/G^*, C^*/G^*, D^*/G^* written so that B^*, C^*, D^*, G^* have no common factor. If G_1 is the greatest common divisor of G^* and $B^*C^* - D^{*2}$, then D^*G^*/G_1, $(B^* - D^*)G^*/G_1$ and $(C^* - D^*)G^*/G_1$ are polynomials with non-negative coefficients.*

THEOREM 7.3. *Let Γ be an RC series-parallel network with impedance functions B^+/Δ^+, C^+/Δ^+, D^+/Δ^+, written so that B^+, C^+, D^+, Δ^+ have no common factor. If Δ_1 is the greatest common divisor of Δ^+ and $B^+C^+ - D^{+2}$, then $D^+\Delta^+/\Delta_1$, $(B^+ - D^+)\Delta^+/\Delta_1$ and $(C^+ - D^+)\Delta^+/\Delta_1$ are polynomials with non-negative coefficients.*

The proof follows. Let Γ be an *RC* series-parallel network for which the network admittance functions Y_{11}, Y_{12}, Y_{22} are specified. According to (2.7), this information determines the ratios of B, C, D to G. Let

$$B = fB^*, \qquad C = fC^*, \qquad D = fD^*, \qquad G = fG^*, \tag{7.1}$$

where B^*, C^*, D^*, G^* have no common factor and are thus determined by the admittance functions up to a multiplicative constant. Suppose that

$$B^*C^* - D^{*2} = G_1H, \qquad G^* = G_1G_2, \tag{7.2}$$

where* G_1 is the greatest common factor of $B^*C^* - D^{*2}$ and G^* so that G_2 and H are relatively prime. Now, from (2.5), (7.1) and (7.2),

$$BC - D^2 = f^2(B^*C^* - D^{*2}) = f^2G_1H = G\Delta = fG_1G_2\Delta,$$

so that, since G_2, H are relatively prime and $fH = G_2\Delta$, it follows that $\Delta = \phi H$ and consequently $f = \phi G_2$, where ϕ is arbitrary. It follows from (7.1) that the most general network functions consistent with the given admittance functions are

$$B = \phi G_2B^*, \qquad C = \phi G_2C^*, \qquad D = \phi G_2D^*, \qquad G = \phi G_1G_2^2, \qquad \Delta = \phi H. \tag{7.3}$$

In a similar manner, if the impedance functions Z_{11}, Z_{12}, Z_{22} are specified for Γ, from (2.6), they determine functions B^+, C^+, D^+, Δ^+ without common factor. Proceeding as above with the roles of Δ and G interchanged, it follows that the most general network functions consistent with the given impedance functions are

$$B = \phi\Delta_2B^+, \qquad C = \phi\Delta_2C^+, \qquad D = \phi\Delta_2D^+, \qquad G = \phi H, \qquad \Delta = \phi\Delta_1\Delta_2^2. \tag{7.4}$$

In these equations ϕ is arbitrary and

$$B^+C^+ - D^{+2} = \Delta_1H, \qquad \Delta^+ = \Delta_1\Delta_2,$$

where Δ_1 is the greatest common factor of $B^+C^+ - D^{+2}$ and Δ^+. In both (7.3) and (7.4),

*An equivalent definition is that G_1 contains all poles of the Y_{ij} where the residue condition is satisfied with equality sign; G_2 contains all other poles.

the greatest common factor of all the networks functions is ϕ. As a result of Theorem 7.1, this proves that D^*G_2, $(B^* - D^*)G_2$, $(C^* - D^*)G_2$ and $D^+\Delta_2$, $(B^+ - D^+)\Delta_2$, $(C^+ - D^+)\Delta_2$ are polynomials with non-negative coefficients. This completes the proof of the theorems.

The preceding discussion may also be used to establish the following theorem:

THEOREM 7.4. *Let Γ be an RC series-parallel network with admittance functions B^*/G^*, C^*/G^*, D^*/G^* and impedance functions B^*/Δ^*, C^*/Δ^*, D^*/Δ^* written so that B^*, C^*, D^*, G^*, Δ^* have no common factor. Then all the coefficients of D^*, $B^* - D^*$ and $C^* - D^*$ must be non-negative.*

The proof follows. For any network Γ, with given admittance (or impedance) functions, the network functions may always be written in the form of (7.3) or, alternatively, of (7.4). Consequently when both the admittance and impedance functions are written in simplest form but subject to the condition that corresponding functions of each kind have the same numerator, we obtain, from (2.6) and (2.7),

$$Y_{11} = \frac{G_2B^*}{G_1G_2^2}, \qquad Y_{22} = \frac{G_2C^*}{G_1G_2^2}, \qquad -Y_{12} = \frac{G_2D^*}{G_1G_2^2},$$

$$Z_{11} = \frac{G_2C^*}{H}, \qquad Z_{22} = \frac{G_2B^*}{H}, \qquad Z_{12} = \frac{G_2D^*}{H}.$$

Since B^*, C^*, D^*, G^* are relatively prime, the only common factor of the Y_{ij} is G_2, but G_2 is relatively prime to H. Hence no further simultaneous reduction in both the Y_{ij}, Z_{ij} is possible. If Γ is an RC series-parallel network, the common numerator G_2D^* of $-Y_{12}$, Z_{12} as well as the differences between the numerators of Y_{11}, Γ_{22} and the numerator of Y_{12} must have non-negative coefficients, in accordance with Theorem 7.2. This proves the theorem.

When the five network functions of Γ have all common factors deleted as in Theorem 7.4, the resulting functions B^*, C^*, D^*, G^*, Δ^* may still have common factors when taken in groups of four. Let B^*, C^*, D^*, G^* and B^*, C^*, D^*, Δ^* have the greatest common factors μ and ν, respectively. Since all five functions are relatively prime, it follows that the pairs μ, ν; μ, Δ^* and ν, G^* are each relatively prime. Then we may write

$$B^* = \mu\nu B^+, \qquad C^* = \mu\nu C^+, \qquad D^* = \mu\nu D^+, \qquad G^* = \mu G_0, \qquad \Delta^* = \nu\Delta_0.$$

From (2.5) and the preceding equations,

$$\mu^2\nu^2(B^+C^+ - D^{+2}) = \mu\nu G_0\nu_0$$

Hence μ divides G_0 and ν divides Δ_0 so that

$$G^* = \mu^2G^+, \qquad \Delta^* = \nu^2\Delta^+, \qquad B^+C^+ - D^{+2} = G^+\Delta^+$$

Theorem 7.4 asserts that the coefficients of $\mu\nu D^+$ are all non-negative. However this need not be true of the reduced numerator of $-Y_{12} = \nu D^+/\mu G^+$ when all three admittance functions Y_{ij} are written in simplest form with common denominator μG^+, nor of the reduced numerator of $Z_{12} = \mu D^+/\nu\Delta^+$ when all three impedance functions Z_{ij} are written in simplest form with common denominator $\nu\Delta^+$. Thus in the illustrative examples of §4, which all refer to RC series-parallel networks, the reduced numerator of Z_{12} has a negative coefficient in example 1 and the reduced numerator of $-Y_{12}$ has

a negative coefficient in example 3. Example 2 illustrates a situation not discussed above in which C^*, D^*, G_1, H but not B^* all have a common factor. When this factor is removed, the common numerator of $-Y_{12}$ and Z_{12} has a negative coefficient.

8. Some realizability questions. We now consider some aspects of the synthesis problem. For a set of admittance functions to be realized by an RC series-parallel network, the previously known residue and coefficient conditions must be satisfied. In addition, the new realizability condition of the preceding section (for example, Theorem 7.2), which is independent of the earlier ones, must be obeyed. But no general realization of admittance functions obeying these conditions exists. Most series-parallel synthesis procedures do little more than describe the necessary extraction of series and parallel components without giving any theoretical basis for when and how this will be possible. The synthesis technique of Ozaki [5] lays such a foundation but is applicable primarily to a restricted class of symmetric RC two-ports. A simple application of Ozaki's method extends the synthesis possibilities as follows:

THEOREM 8.1. *Let Y_{11}, Y_{22} be two RC admittances and let $-Y_{12} = D^*/G^*$ be the quotient of two polynomials without common factors, where all the coefficients of D^* are non-negative, $Y_{12}(0) = 0$ if $Y_{11}(0)Y_{22}(0) = 0$, and each pole of Y_{12} (possibly, including ∞) is also a pole of both Y_{11} and Y_{22}. Then an RC series-parallel network exists which realizes Y_{11}, KY_{12}, Y_{22} for all sufficiently small positive values of K.*

To prove this result, we first decompose the Y_{ij} in any manner into a sum

$$Y_{ij} = \sum_k {}_{(k)}Y_{ij},$$

such that all the numerators of the ${}_{(k)}Y_{12}$ are Hurwitz polynomials and each pole of ${}_{(k)}Y_{12}$ is also a pole of ${}_{(k)}Y_{11}$ and ${}_{(k)}Y_{22}$. This decomposition is always possible. Thus one possibility, usually not the best, is the following: If

$$-Y_{12} = \frac{D^*}{G^*} = \frac{\sum_{h=0}^{m} D_h s^h}{G^*},$$

let

$$-{}_{(k)}Y_{12} = D_k s^k/G^*, \qquad {}_{(k)}Y_{11} = Y_{11}/(m+1), \qquad {}_{(k)}Y_{22} = Y_{22}/(m+1),$$
$$k = 0, 1, 2, \cdots, m.$$

Having chosen the ${}_{(k)}Y_{ij}$, expand each ${}_{(k)}Y_{ij}$ so as to exhibit its residues:

$${}_{(k)}Y_{11} = {}_{(k)}a_0 + {}_{(k)}a_\infty s + \sum_{l=1}^{n_k} \frac{{}_{(k)}a_l s}{s + {}_{(k)}\delta_l},$$

$${}_{(k)}Y_{12} = {}_{(k)}b_0 + {}_{(k)}b_\infty s + \sum_{l=1}^{n_k} \frac{{}_{(k)}b_l s}{s + {}_{(k)}\delta_l},$$

$${}_{(k)}Y_{22} = {}_{(k)}c_0 + {}_{(k)}c_\infty s + \sum_{l=1}^{n_k} \frac{{}_{(k)}c_l s}{s + {}_{(k)}\delta_l}.$$

Choose K_0 so small so that, for each value of k,

$${}_{(k)}a_0 \geq K_0 {}_{(k)}b_0, \qquad {}_{(k)}c_0 \geq K_0 {}_{(k)}b_0, \qquad {}_{(k)}a_\infty \geq K_0 {}_{(k)}b_\infty, \qquad {}_{(k)}c_\infty \geq K_0 {}_{(k)}b_\infty,$$
$${}_{(k)}a_l \geq K_0 \,|{}_{(k)}b_l|, \qquad {}_{(k)}c_l \geq K_0 \,|{}_{(k)}b_l|, \qquad l = 1, 2, \cdots, n_k$$

This is possible since, according to the hypothesis of the theorem, if any of the left members of the inequalities are zero, the corresponding right members are also. Now write

$$\frac{1}{K_0}\,{}_{(k)}Y = {}_{(k)}b_0 + {}_{(k)}b_\infty s + \sum_{l=1}^{n_k} \frac{|{}_{(k)}b_l|\, s}{s + {}_{(k)}\delta_l},$$

$${}_{(k)}Y_1 = {}_{(k)}Y_{11} - {}_{(k)}Y, \qquad {}_{(k)}Y_2 = {}_{(k)}Y_{22} - {}_{(k)}Y,$$

where, according to the preceding inequalities ${}_{(k)}Y_1$, ${}_{(k)}Y_2$ are RC admittances. Now use the method of Ozaki to synthesize the admittance functions

$$Y'_{11} = Y'_{22} = {}_{(k)}Y, \qquad Y'_{12} = K_0\,{}_{(k)}Y_{12},$$

and modify the resulting network by adding shunt admittances ${}_{(k)}Y_1$ between nodes 1 and 0 and ${}_{(k)}Y_2$ between nodes 2 and 0 to form an RC series-parallel network Γ_k. Then Γ_k realizes the admittance functions ${}_{(k)}Y_{11}$, ${}_{(k)}Y_{22}$, $K_0\,{}_{(k)}Y_{12}$. The parallel connection Γ of the Γ_k then realizes the admittance functions Y_{11}, Y_{22}, K_0Y_{12}.

To realize Y_{11}, Y_{22}, KY_{12}, where $K \leqq K_0$, we proceed as follows*: Change the admittance level in Γ by the factor K/K_0 to produce Γ' whose admittance functions are KY_{11}/K_0, KY_{22}/K_0, KY_{12}. Modify Γ' by introducing two new branches joining nodes 0 and 1 and nodes 0 and 2 respectively, whose admittances are $(1 - K/K_0)Y_{11}$ and $(1 - K/K_0)Y_{22}$ respectively. Then the admittance functions of the modified network are Y_{11}, Y_{22}, KY_{12}. This completes the proof of Theorem 8.1.

In this synthesis, there is no theoretical determination of the maximum value K_0 which can be realized. A more serious defect in the theory is the lack of a complete description of all the possible D^*'s which are realizable by series-parallel networks by some method. The difficulties associated with this problem arise from the following circumstances. Series-parallel networks exist whose admittance functions Y_{ij}, specified as in (7.1), are such that the reduced numerator D^* of $-Y_{12}$ has negative coefficients (even if, as demanded by Theorem 7.2, the coefficients of D^*G_2 are non-negative). Example 3 of §4 is such a network. For admittance functions of this kind, there is no general series-parallel synthesis technique and consequently we cannot decide apriori whether such functions can or cannot be realized by series-parallel networks.

Actually, not all admittance functions which obey the residue and coefficient conditions and also the new conditions of §7 (such as Theorem 7.2) can be synthesized by series-parallel networks. That is, all the preceding conditions are necessary but not sufficient for RC series-parallel realizability. We now give two illustrations of this fact, omitting the relevant proofs which will appear elsewhere.

Example 1.

$$Y_{11} = a_0 + a_\infty s + \frac{as}{s+1}, \qquad Y_{22} = c_0 + c_\infty s + \frac{cs}{s+1}, \qquad -Y_{12} = b_0 + b_\infty s - \frac{bs}{s+1}$$

where all the letters are positive. It may be shown that all such Y_{ij} which satisfy the residue and coefficient conditions may be synthesized *if and only if* the numerator of $-Y_{12}$ has non-negative coefficients. This last condition is equivalent to

$$0 \leqq b \leqq b_0 + b_\infty .$$

*An alternative method is to replace K_0 by K throughout the preceding synthesis.

However, if the pole $s = -1$ is not compact, Theorem 7.2 requires that $-(s+1)Y_{12}$ have non-negative coefficients or, in equivalent terms, that

$$0 < b \leqq b_0 + b_\infty + \operatorname{Min}(b_0, b_\infty).$$

Thus values of b in the range

$$b_0 + b_\infty < b \leqq b_0 + b_\infty + \operatorname{Min}(b_0, b_\infty),$$

generate admittance functions Y_{ij} (having a non-compact pole at $s = -1$) which satisfy all the known necessary conditions but cannot be synthesized by a series-parallel network.

Example 2.

$$-Y_{12} = \frac{b_0 s^3 + b_3}{(s+\gamma_1)(s+\gamma_2)}, \quad Y_{11} = \frac{b_0 s^3 + a_1 s^2 + a_2 s + b_3}{(s+\gamma_1)(s+\gamma_2)}, \quad Y_{22} = \frac{b_0 s^3 + c_1 s^2 + c_2 s + b_3}{(s+\gamma_1)(s+\gamma_2)}$$

where all the letters are positive and a_1, a_2, c_1, c_2 are chosen so that Y_{11}, Y_{22} are RC admittances, all the residue conditions are satisfied with equality signs and the coefficient conditions are obeyed. Since $b_0 > 0$, $b_3 > 0$, Theorem 7.2 is also true. Nevertheless, it can be proved that these Y_{ij} cannot be realized by any RC series-parallel network.

The preceding discussion proves that Theorem 7.2, taken in conjunction with the previously known residue and coefficient conditions still does not constitute a set of sufficient conditions for series-parallel synthesis. These examples suggest an additional problem. Do RC admittance functions exist, whose synthesis demands a more general structure than the series-parallel structure? To date, no counter examples are known to the ten year old conjecture of Darlington [2, p. 295] that every realizable set of RC Y_{ij} may be synthesized by a series-parallel network. But neither has the validity of the Darlington conjecture been established. A proof of the truth or falsity of this conjecture may depend upon the new necessary property of series-parallel networks.

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