A FREE BOUNDARY PROBLEM FOR THE HEAT EQUATION WITH PRESCRIBED FLUX AT BOTH FIXED FACE AND MELTING INTERFACE*

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1. Introduction and summary. The following is typical of problems that have been considered by many writers in recent years, e.g., Friedman [4, 5], Kyner [8, 9], Miranker [11], Douglas [3], and Kolodner [7]. A semi-infinite slab occupies the region $0 \le x < \infty$, where initially the region $x > a$ is solid at the melting temperature 0 and the region $0 \le x \le a$ is liquid at prescribed initial temperature $\phi(x) \ge 0$, with $\phi(a) = 0$. The heat flux $f(t) \le 0$ at the fixed face $x = 0$ is prescribed. We wish to determine the temperature distribution $u(x, t)$ in the liquid and the position $s(t)$ of the melting interface. The full statement of the problem is

$$\begin{gathered} \kappa u_{xx} = u_t\,, \quad 0 < x < s(t); \qquad u(x, 0) = \phi(x), \quad \phi(a) = 0, \\ k u_x(0, t) = f(t), \quad u(s(t), t) = 0, \quad -\rho l s'(t) + k u_x(s(t), t) = 0, \quad s(0) = a, \end{gathered} \tag{1}$$

where k is the thermal conductivity, κ the thermal diffusivity, ρ the density of both liquid and solid, and l the latent heat of fusion. An existence and uniqueness theorem for (1) has been given by Friedman [5] and Kyner [9] in the case $a = 0$, and by Friedman [4I], Kyner [8], and Miranker [11] in the case $a > 0$. This theorem (e.g., [4I, p. 515] or [6, p. 225]) states that (1) has a unique solution, that $s(t)$ is nondecreasing, and that $s(t)$ is strictly increasing if $\phi(x) \not\equiv 0$ or $f(t) \not\equiv 0$ in any interval $0 < t < T$. The definition of solution and the regularity conditions satisfied by $\phi(x)$ and $f(t)$ are given in [4I, p. 501].

Another problem, considered by Landau [10], Boley and Weiner [1], Citron [2], and Sherman [13], is the following: a slab initially occupies the region $0 \le x \le a$ and has initial temperature $\phi(x) \le 0$, $\phi(a) = 0$, where 0 is the melting temperature. It is insulated at $x = 0$ and has heat flux $q(t) \ge 0$ at the opposite face. If melted material is formed it is removed immediately so that the heat input is always at the melting interface. We want to determine $u(x, t)$ and $s(t)$. The full statement of this problem is (subject to the comments below)

$$\begin{gathered} \kappa u_{xx} = u_t\,, \quad 0 < x < s(t); \qquad u(x, 0) = \phi(x), \quad \phi(a) = 0, \\ u_x(0, t) = 0, \quad u(s(t), t) = 0, \quad -\rho l s'(t) + k u_x(s(t), t) = q(t), \quad s(0) = a. \end{gathered} \tag{2}$$

It has been pointed out in [10] and [13] that (2) is not the precise mathematical formulation of the problem described above. For if, over some positive time interval, the heat input $q(t)$ is insufficient to maintain melting we must drop the condition $u(s(t), t) = 0$ for that time interval and deal with a heat conduction problem in a fixed slab with prescribed fluxes at both faces. We will discuss later the appropriate formulation of the problem described above. Referring to the problem discussed in the first paragraph it is clear that (2) refers to a semi-infinite slab which is initially liquid at the melting temperature 0

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in the region $x > a$, solid in the region $0 \leq x \leq a$ with initial temperature $\phi(x)$, insulated at $x = 0$, and having flux $q(t)$ at the melting interface. In [13] the author proved an existence theorem for (2) for small t under the hypothesis that

$$\rho l s'(0) = -q(0) + k\phi'(a) < 0. \tag{3}$$

It is proved that $u(x, t)$ and $s(t)$ exist and that $s'(t) < 0$ for small t. Thus if the interface is already moving to the left, i.e., if melting has begun, then it will continue to do so for a short time thereafter. However the statement in [13] that a uniqueness theorem has been proved is not true since a discussion parallel to Friedman's [4I, p. 511] is needed. This discussion is given in Sec. 3 of this paper for the more general problem (4).

If in (1) we change the sign of $\phi(x)$ and $f(t)$, so that $\phi(x) \leq 0$ and $f(t) \geq 0$, we obtain, so to speak, the mirror image of the problem stated in the first paragraph, namely, a semi-infinite slab initially liquid at melting temperature 0 in $x > a$, solid in $0 \leq x \leq a$ at initial temperature $\phi(x) \leq 0$, and having heat flux $f(t) \geq 0$ at $x = 0$. Clearly if u is a solution of (1) then $-u$ is a solution of the problem just stated. Accordingly we may apply the theorem already proved for (1) and state that $u(x, t)$ and $s(t)$ exist for all $t > 0$ and that $s(t)$ is nondecreasing. Furthermore $s(t)$ is strictly increasing if $\phi(x)$ is not identically zero or if $f(t)$ is not identically zero in any interval $0 < t < T$.

It is clear that we may give a unified treatment of (1) and (2) by considering the problem

$$\begin{gathered} \kappa u_{xx} = u_t\,, \quad 0 < x < s(t); \qquad u(x, 0) = \phi(x), \quad \phi(a) = 0, \\ k u_x(0, t) = f(t), \quad u(s(t), t) = 0, \quad -\rho l s'(t) + k u_x(s(t), t) = q(t), \quad s(0) = a, \end{gathered} \tag{4}$$

where $\phi(x) \leq 0$, $f(t) \geq 0$, and $q(t) \geq 0$. Thus we are dealing with a semi-infinite slab which is initially liquid at the melting temperature 0 in the region $x > a$, solid in the region $0 \leq x \leq a$ with initial temperature distribution $\phi(x)$, heat withdrawn at the rate $f(t)$ at the fixed face $x = 0$ and added at the rate $q(t)$ at the moving interface $x = s(t)$. We get problem (1) by setting $q(t) = 0$ and replacing $\phi(x)$ and $f(t)$ by $-\phi(x)$ and $-f(t)$. We get problem (2) by setting $f(t) = 0$. In problem (1) $s(t)$ is nondecreasing but in (4) it is clear that $s(t)$ does not necessarily have monotonic behaviour. We observe also that it is possible that the interface reaches $x = 0$ at some finite time σ. This is clear from the example $\phi(x) = 0$, $f(t) = 0$. The solution of (4) is $u(x, t) = 0$ and

$$s(t) = \left[a\rho l - \int_0^t q(\tau)\, d\tau \right] \Big/ \rho l. \tag{5}$$

The interface reaches $x = 0$ at $t = \sigma$ if the numerator of (5) is 0 for $t = \sigma$. We note that the interface need not move at all even though the heat input $q(t)$ is positive. Consider, for example, a fixed slab occupying the region $0 \leq x \leq a$ with flux $f(t) = 0$ at $x = 0$, temperature 0 (the melting temperature) at $x = a$ for all t, and initial temperature $\phi(x) \leq 0$. If $u^1(x, t)$ is the solution of this problem then problem (4), with the same $f(t) = 0$ and $\phi(x)$ and with $q(t) = k u_x^1(a, t)$, will have $s(t) = a$. That $u_x^1(a, t) > 0$ can be seen as follows: we extend the solution to $-a < x < a$, $t > 0$ by taking $\phi(-x) = \phi(x)$ and $u^1(-a, t) = 0$. Then $u^1(x, t) < 0$ in $-a < x < a$ by the strong maximum principle [12], [6, Chapter 2] (assuming $\phi(x)$ is not identically zero) and $u_x^1(a, t) > 0$ by a lemma of Friedman [6, p. 49].

We give now a precise statement of the theorem covering the existence and uniqueness of the solution of (4). For simplicity of writing we choose units so that $\kappa = 1$. We will also write $\alpha = (\rho l)^{-1}$. Following Friedman we define a solution as follows:

Definition. $u(x, t)$, $s(t) > 0$ is a solution of (4) for $0 < t < T$, where $0 < T \leq \infty$, if (a) u_{xx} and u_t are continuous for $0 < x < s(t)$, $0 < t < T$, (b) u and u_x are continuous for $0 \leq x \leq s(t)$, $0 < t < T$, (c) u is continuous also for $t = 0$, $0 < x \leq a$, and

$$-\infty < \liminf u(x, t) \leq \limsup u(x, t) \leq 0$$

as $(x, t) \to (0, 0)$, (d) $s'(t)$ exists and is continuous on $0 \leq t < T$, (e) system (4) is satisfied.

We suppose that $f(t)$ and $q(t)$ are defined over $0 \leq t < \infty$ and that $\phi(x)$ is defined over $0 \leq x \leq a$. If a solution of (4) exists we may integrate $u_{xx} - u_t = 0$ over the region in the (x, t) plane bounded by $x = 0$, $x = s(t)$, and the horizontal lines at 0 and t. We get

$$s(t) = a + \alpha \int_0^t [f(\tau) - q(\tau)]\, d\tau - k\alpha \int_0^a \phi(x)\, dx + k\alpha \int_0^{s(t)} u(x, t)\, dx. \tag{6}$$

It is plausible that $u(x, t) \leq 0$, and this is proved in Sec. 4. Complete melting has occurred at time $t = \sigma$, or the interface has reached $x = 0$ at time $t = \sigma$, if and only if $s(\sigma) = 0$ and there is no smaller value of t such that $s(t) = 0$. From (6) it is clear that $s(\sigma) = 0$ is equivalent to the vanishing of the sum of the first three terms on the right of (6). Accordingly we define σ to be the first value of t for which

$$a + \alpha \int_0^\sigma [f(\tau) - q(\tau)]\, d\tau - k\alpha \int_0^a \phi(x)\, dx = 0,$$

and $\sigma = \infty$ if no such value exists. We may now state the theorem.

Theorem. Suppose $\phi(x)$ has a continuous derivative and that $f(t)$ and $q(t)$ are continuous. Then for any $T \leq \sigma$ there is a unique solution of (4). If $\sigma < \infty$ then $s(\sigma) = 0$.

The proof of the theorem, following the method of Friedman [4I], is given in Secs. 2, 3, and 4. In Sec. 2 we show that problem (4) is equivalent to the integral equation $v = Sv$ given by (11a); here $v(t) = u_x(s(t), t)$ and the $s(t)$ appearing in Sv is given by (11b). In Sec. 3 we define an appropriate complete metric space and show that $w = Sv$ is a contracting mapping of that space into itself. We establish thereby an existence and uniqueness theorem for $v = Sv$ for small t. In Sec. 4 we extend the local theorem to arbitrary t.

The method of proof used in [13] may be applied here but it is not as concise as Friedman's. Both methods use the contracting mapping principle. Kyner [9] uses the Schauder fixed point theorem for problem (1) but the method does not seem to be easy to apply to (4).

Returning to the problem in which the fixed face is insulated and the heat flux is always at the melting interface, i.e., we remove instantaneously the melted material, the appropriate formulation seems to be the following:

$$\begin{gathered}\kappa u_{xx} = u_t\,, \quad 0 < x < s(t); \qquad u(x, 0) = \phi(x) \leq 0, \quad u_x(0, t) = 0, \\ -\rho l s'(t) + k\phi_x(s(t), t) = q(t) \geq 0, \quad s(0) = a, \\ s'(t) \leq 0, \quad u(s(t), t) \leq 0, \quad u(s(t), t)s'(t) = 0.\end{gathered} \tag{7}$$

We note the differences between (7) and (2). In (7) we require $s(t)$ to be nonincreasing, and the condition $u(s(t), t) = 0$ applies only when $s(t)$ is strictly decreasing. When $s(t)$

is constant, so that there is no melting, we require only that $u(s(t), t)$ be less than or equal to the melting temperature 0. The condition $\phi(a) = 0$, which appears in (2), is omitted. A possible approach to (7) is the following. If $\phi(a) < 0$ we take $s(t) = a$ and solve the fixed boundary problem (7) to the first time t_0 for which $u(a, t_0) = 0$. If there is no such t_0 then (7) is completely solved to $t = \infty$. We may therefore assume $\phi(a) = 0$. If we consider the solution of (4) corresponding to the given boundary and initial data of (7) then, in the vicinity of $t = 0$, we may have $s(t)$ decreasing, or increasing, or oscillating infinitely often about a. If $s(t)$ is decreasing then (7) is satisfied by this solution of (4) in the vicinity of $t = 0$. If $s(t)$ is increasing then it seems plausible to take $s(t) = a$ in the vicinity of $t = 0$ and then solve (7) as a fixed boundary problem. It seems reasonable to expect that $u(a, t) \leq 0$. But if $s(t)$ oscillates infinitely often about a then it is not clear that (7) has a solution.

If we let $t \to 0$ in the second free boundary condition in (4) we get $s'(0) = -\alpha q(0) + k\alpha u_x(s(0), 0)$. It is easily seen from (11a) that $u_x(s(0), 0) = v(0) = \phi'(a)$. Thus $s'(0) = -\alpha q(0) + k\alpha\phi'(a)$. Since $s'(t)$ is continuous the interface moves to the right or the left in the vicinity of $t = 0$ according as $-q(0) + k\phi'(a)$ is positive or negative.

2. Reduction of (4) to an equivalent integral equation. Let

$$K(x, t; \xi, \tau) = [4\pi(t - \tau)]^{-1/2} \exp [-(x - \xi)^2/4(t - \tau)],$$
$$G(x, t; \xi, \tau) = K(x, t; \xi, \tau) - K(-x, t; \xi, \tau),$$
$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau).$$

Let $u(x, t)$, $s(t)$ satisfy (4) over $0 \leq t < T$. Then, integrating the identity

$$\frac{\partial}{\partial \xi}(Nu_\xi - uN_\xi) - \frac{\partial}{\partial \tau}(Nu) = 0 \tag{8}$$

over the domain $0 < \xi < s(\tau)$, $\epsilon < \tau < t - \epsilon$, using Green's theorem and letting $\epsilon \to 0$ we get

$$u(x, t) = \int_0^t u_\xi(s(\tau), \tau)N(x, t; s(\tau), \tau)\, d\tau - k^{-1} \int_0^t f(\tau)N(x, t; 0, \tau)\, d\tau + \int_0^a \phi(\xi)N(x, t; \xi, 0)\, d\xi. \tag{9}$$

Taking derivatives with respect to x on both sides of (9) and letting $x \to s(t)$ we get, using the lemma in [4I, p. 501] and writing $v(t) = u_x(s(t), t)$,

$$v(t) = 2 \int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau)\, d\tau - 2k^{-1} \int_0^t f(\tau)N_x(s(t), t; 0, \tau)\, d\tau + 2 \int_0^a \phi(\xi)N_x(s(t), t; \xi, 0)\, d\xi. \tag{10}$$

Since $N_x = -G_\xi$ we may use partial integration on the last integral:

$$v(t) = 2 \int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau)\, d\tau - 2k^{-1} \int_0^t f(\tau)N_x(s(t), t; 0, \tau)\, d\tau + 2 \int_0^a \phi'(\xi)G(s(t), t; \xi, 0)\, d\xi. \tag{11a}$$

We have also the equation

$$s(t) = a - \alpha \int_0^t q(\tau)\, d\tau + k\alpha \int_0^t v(\tau)\, d\tau. \tag{11b}$$

Thus if $u(x, t)$ and $s(t)$ satisfy (4) over $0 \leq t < T$ then $v(t)$ and $s(t)$ satisfy (11a, b). Conversely, suppose $v(t)$ is a continuous solution of (11a) over $0 \leq t < T$, where $s(t)$ is given by (11b) and is positive. We may replace $u_\xi(s(\tau), \tau)$ in the first integrand in (9) by $v(\tau)$ and thereby define $u(x, t)$; with this replacement we will write (9)*. We will show that this $u(x, t)$, together with $s(t)$, satisfy (4). It is easily seen from (9)* that $u_{xx} = u_t$ in $0 < x < s(t)$, $0 < t < T$. Letting $t \to 0$ in (9)* we get $u(x, 0) = \phi(x)$. If we form $u_x(x, t)$ and let $x \to s(t)$ we find that $u_x(s(t), t)$ is $v(t)/2$ plus half the right side of (10), so that $u_x(s(t), t) = v(t)$. Differentiating (11b) we get the relation $k\alpha u_x(s(t), t) = s'(t) + \alpha q(t)$. If we let $x \to 0$ in $u_x(x, t)$ then it is easily proved that $u_x(0, t) = k^{-1}f(t)$. Thus we have only to prove that $u(s(t), t) = 0$. If we introduce $u(\xi, \tau)$, defined by (9)*, into the identity (8), integrate over $0 < \xi < s(\tau)$, $\epsilon < \tau < t - \epsilon$, use Green's theorem and the boundary properties of u just proved then, letting $\epsilon \to 0$, we get

$$\int_0^t u(s(\tau), \tau)\{N(x, t; s(\tau), \tau)s'(\tau) + N_\xi(x, t; s(\tau), \tau)\}\, d\tau = 0. \tag{12}$$

Letting $x \to s(t)$ in (12) we get

$$-\tfrac{1}{2}u(s(t), t) + \int_0^t u(s(\tau), \tau)\{N(s(t), t; s(\tau), \tau)s'(\tau) + N_\xi(s(t), t; s(\tau), \tau)\}\, d\tau = 0. \tag{13}$$

The expression in the braces in (13) has absolute value $\leq c(t - \tau)^{-1/2}$, where c is a constant depending on t. Thus if $g(t) = |u(s(t), t)|$ then

$$\begin{aligned} g(t) &\leq 2c \int_0^t g(\tau)(t - \tau)^{-1/2}\, d\tau \\ &\leq 4c^2 \int_0^t \int_0^\tau g(\zeta)[(\tau - \zeta)(t - \tau)]^{-1/2}\, d\zeta\, d\tau = 4c^2\pi \int_0^t g(\zeta)\, d\zeta. \end{aligned} \tag{14}$$

It is easily proved from (14) that $g(t) \equiv 0$. Thus (4) is equivalent to (11a, b) and it is therefore sufficient to prove the existence and uniqueness of a solution of (11a, b).

3. Local existence and uniqueness of a solution of (11a, b). Let $C(T)$ be the Banach space of continuous functions $v(t)$ on $0 \leq t \leq T$ with the norm $||v|| = \max |v(t)|$. Let $C(T, M)$ be the closed sphere $||v|| \leq M$. If we write $v = Sv$ for (11a) then $w = Sv$, with $s(t)$ defined by (11b), defines a mapping of $C(T, M)$ into $C(T)$ which, for appropriate choice of T and M, becomes a contracting mapping of $C(T, M)$ into itself. To prove this we note that

$$|s(t) - s(\tau)| \leq \alpha(||q|| + kM)(t - \tau) = m(t - \tau). \tag{15}$$

We select M and T subject to

$$2mT = 2\alpha(||q|| + kM)T \leq a. \tag{16}$$

Then

$$a/2 \leq s(t) \leq 3a/2, \quad 0 \leq t \leq T, \tag{17}$$

and

$$|w(t)| \leq ||v|| \int_0^t |[s(t) - s(\tau)]/(t - \tau)| \, K(s(t), t; s(\tau), \tau) \, d\tau$$

$$+ ||v|| \int_0^t \{[s(t) + s(\tau)]/(t - \tau)\} K(-s(t), t; s(\tau), \tau) \, d\tau$$

$$+ 2k^{-1} ||f|| \int_0^t [s(t)/(t - \tau)] K(s(t), t; 0, \tau) \, d\tau$$

$$+ \left| 2 \int_0^a \phi'(\xi) G(s(t), t; \xi, 0) \, d\xi - \phi'(a) \right| + |\phi'(a)|. \tag{18}$$

The first term on the right of (18) is less than or equal to

$$Mm(4\pi)^{-1/2} \int_0^t (t - \tau)^{-1/2} \, d\tau = Mm(t/\pi)^{1/2} \leq Mm(T/\pi)^{1/2}.$$

The second term on the right of (17) is less than or equal to

$$M \int_0^t [3a/(t - \tau)] K(a, t; 0, \tau) \, d\tau$$

$$= (3aM/2\pi^{1/2}) \int_{t^{-1}}^{\infty} x^{-1/2} \exp(-a^2 x/4) \, dx = MB_1(t) \leq MB_1(T),$$

and by a similar argument we can show that the third term on the right in (18) is $\leq k^{-1} ||f|| \, B_2(T)$, where $B_2(T)$ is $B_1(T)$ with $\exp(-a^2x/4)$ replaced by $\exp(-a^2x/16)$. $B_1(T)$ and $B_2(T)$ both go to 0 as $T \to 0$. Let $A(T)$ be the maximum of the fourth term on the right of (18) over $0 \leq t \leq T$. Since that term goes to 0 as $t \to 0$ we have $A(T) \to 0$ as $T \to 0$. Thus

$$||w|| \leq M\alpha(||q|| + kM)(T/\pi)^{1/2} + MB_1(T) + k^{-1} ||f|| \, B_2(T) + A(T) + |\phi'(a)|. \tag{19}$$

We choose now $M = |\phi'(a)| + 1$ and choose T so that (16) is satisfied and also the inequality

$$M\alpha(||q|| + kM)(T/\pi)^{1/2} + MB_1(T) + k^{-1} ||f|| \, B_2(T) + A(T) \leq 1. \tag{20}$$

Then $||w|| \leq M$ and therefore S maps $C(T, M)$ into itself.

Next we show that on appropriately choosing T and M the mapping S is a contraction. Let $w = Sv$, $w^* = Sv^*$ and let $||v - v^*|| = \epsilon$. Then $\epsilon \leq 2M$, from (11b) $|s(t) - s^*(t)| \leq t\epsilon$, and (15) is valid for s and s^*. Imposing condition (16), (17) is valid for s and s^*. We write

$$w - w^* = 2 \int_0^t \{v(\tau) N_x(s(t), t; s(\tau), \tau) - v^*(\tau) N_x(s^*(t), t; s^*(\tau), \tau)\} \, d\tau$$

$$- 2k^{-1} \int_0^t f(\tau) \{N_x(s(t), t; 0, \tau) - N_x(s^*(t), t; 0, \tau)\} \, d\tau \tag{21}$$

$$+ 2 \int_0^a \phi'(\xi) \{G(s(t), t; \xi, 0) - G(s^*(t), t; \xi, 0)\} \, d\xi.$$

Consider the third term on the right of (21). Using the mean value theorem there is a $\sigma(t)$ between $s(t)$ and $s^*(t)$ such that

$$\left|2 \int_0^a \phi'(\xi)\{K(s(t), t; \xi, 0) - K(s^*(t), t; \xi, 0)\} d\xi\right|$$
$$\leq 2 \|\phi'\| \int_0^a |s(t) - s^*(t)| \, |(\sigma(t) - \xi)/2t| \, K(\sigma(t), t; \xi, 0) \, d\xi$$
$$\leq \|\phi'\| \, t\epsilon(\pi t)^{-1/2} \int_0^a |(\sigma(t) - \xi)/2t| \exp(-(\sigma(t) - \xi)^2/4t) \, d\xi. \tag{22}$$

On introducing $x = (\sigma(t) - \xi)/(2t)^{1/2}$ the integral on the right side of (22) is seen to be less than 2. Thus the left side of (22) is $\leq 2 \|\phi'\| \epsilon(t/\pi)^{1/2} \leq 2 \|\phi'\| \epsilon(T/\pi)^{1/2}$. If we now consider the left side of (22) with $s(t)$ and $s^*(t)$ replaced by $-s(t)$ and $-s^*(t)$ we derive again the same upper bound. Thus the absolute value of the third term on the right of (21) does not exceed $4 \|\phi'\| \epsilon(T/\pi)^{1/2}$.

Considering now the second term on the right of (21) we note that its absolute value is

$$4k^{-1} \left|\int_0^t f(\tau)\{K_x(s(t), t; 0, \tau) - K_x(s^*(t), t; 0, \tau)\} d\tau\right| \tag{23}$$

and by the mean value theorem (23) is equal to

$$4k^{-1} \left|\int_0^t f(\tau)(s(t) - s^*(t))K_{xx}(\sigma(t), t; 0, \tau) \, d\tau\right|$$
$$\leq 4k^{-1} \|f\| \, t\epsilon \int_0^t |\{\sigma^2(t)/4(t - \tau)^2\} - \{2(t - \tau)\}^{-1}| \, K(\sigma(t), t; 0, \tau) \, d\tau. \tag{24}$$

Using (17), which is also true for $\sigma(t)$, and introducing $x = (t - \tau)^{-1}$ the integral on the right of (24) is less than

$$\int_0^t [\{9a^2/16(t - \tau)^2\} + \{2(t - \tau)\}^{-1}]K(a/2, t; 0, \tau) \, d\tau$$
$$= \int_{t^{-1}}^\infty [(9a^2x^{1/2}/32\pi^{1/2}) + (16\pi x)^{-1/2}] \exp(-a^2x/16) \, dx = B_3(t).$$

Since $B_3(t) < B_3(T)$ we see that absolute value of the second term on the right of (21) is $\leq 4k^{-1} \|f\| \epsilon T B_3(T)$. We note that $B_3(T) \to 0$ as $T \to 0$.

In the discussion of the first term on the right of (21) we write N_x for $N_x(s(t), t; s(\tau), \tau)$ with analogous meaning for K_x and K. We write N_x^*, K_x^* and K^* if $s^*(t)$ and $s^*(\tau)$ replace $s(t)$ and $s(\tau)$. We write K_x^- and K^- if $-s(t)$ replaces $s(t)$ and K_x^{-*} and K^{-*} for the replacement of $s(t)$ and $s(\tau)$ by $-s^*(t)$ and $s^*(\tau)$. The first term on the right of (21) may be written

$$2 \int_0^t [v(\tau) - v^*(\tau)]N_x \, d\tau + 2 \int_0^t v^*(\tau)(N_x - N_x^*) \, d\tau. \tag{25}$$

The absolute value of the first term in (25) is, using estimates obtained for the right side of (18), less than or equal to

$$2\epsilon \int_0^t |N_x| \, d\tau \leq 2\epsilon \int_0^t (|K_x| + |K_x^-|) \, d\tau \leq 2\epsilon[m(T/\pi)^{1/2} + B_1(T)].$$

The absolute value of the second term in (25) is less than or equal to

$$2M \int_0^t |K_x - K_x^*| \, d\tau + 2M \int_0^t |K_x^- - K_x^{-*}| \, d\tau. \tag{26}$$

We may write

$$K_x - K_x^* = [2(t - \tau)]^{-1}\{K([s(t) - s(\tau)] - [s^*(t) - s^*(\tau)]) + (K - K^*)[s^*(t) - s^*(\tau)]\}$$
$$= K[2(t - \tau)]^{-1}\left\{k\alpha \int_\tau^t [v(\xi) - v^*(\xi)]\, d\xi + [s^*(t) - s^*(\tau)][1 - \exp f(t, \tau)]\right\} \tag{27}$$

where

$$f(t, \tau) = [4(t - \tau)]^{-1}([s(t) - s(\tau)]^2 - [s^*(t) - s^*(\tau)]^2)$$
$$= [4(t - \tau)]^{-1}\left[k\alpha \int_\tau^t [v(\xi) - v^*(\xi)]\, d\xi\right][s(t) - s(\tau) + s^*(t) - s^*(\tau)].$$

Thus we have $|f(t, \tau)| \leq k\alpha\epsilon mT/2$. Since $\epsilon \leq 2M$ we have also $|f(t, \tau)| \leq k\alpha mMT$. Now we impose the condition

$$k\alpha mMT = k\alpha^2(||q|| + kM)MT \leq 1 \tag{28}$$

and using $|1 - e^x| \leq 2\,|x|$ for $|x| \leq 1$ we get from (27)

$$|K_x - K_x^*| \leq |K|\, 2^{-1}(k\alpha\epsilon + k\alpha\epsilon m^2T) \leq k\alpha\epsilon(16\pi(t - \tau))^{-1/2}(1 + m^2T).$$

Then the first term in (26) is $\leq Mk\alpha\epsilon(T/\pi)^{1/2}(1 + m^2T)$. Using the mean value theorem we may write

$$K_x^- - K_x^{-*} = \{s(t) + s(\tau) - [s^*(t) + s^*(\tau)]\}$$
$$\cdot\{[g(t, \tau)/2(t - \tau)]^2 - [2(t - \tau)]^{-1}\}K(g(t, \tau), t; 0, \tau),$$

where $g(t, \tau)$ lies between $s(t) + s(\tau)$ and $s^*(t) + s^*(\tau)$. Then $g(t, \tau)$ lies between a and 3a. Using

$$|s(t) - s^*(t)| = k\alpha \left|\int_0^t [v(\xi) - v^*(\xi)]\, d\xi\right| \leq k\alpha T\epsilon$$

the second term of (26) is less than or equal to

$$4k\alpha MT\epsilon \int_0^t \{[3a/2(t - \tau)]^2 + [2(t - \tau)]^{-1}\}K(a, t; 0, \tau)\, d\tau$$
$$\leq 2k\alpha MT\epsilon\pi^{-1/2} \int_{t^{-1}}^\infty [(9a^2/4)x^{1/2} + (4x)^{-1/2}] \exp(-a^2x/4)\, dx$$
$$= MT\epsilon B_4(t) \leq MT\epsilon B_4(T). \tag{29}$$

$B_4(T) \to 0$ as $T \to 0$.

Collecting the results we get $||w - w^*|| \leq \epsilon F(M, T)$, where

$$F(M, T) = 4\,||\phi'||\,(T/\pi)^{1/2} + 4k^{-1}\,||f||\,TB_3(T) + 2m(T/\pi)^{1/2} + 2B_1(T) + k\alpha M(T/\pi)^{1/2}(1 + m^2T) + MTB_4(T).$$

Thus if we choose $M = |\phi'(a)| + 1$ and T subject to (16), (20), (28) and $F(M, T) < 1$ then $w = Sv$ is a contracting mapping of $C(T, M)$ into itself. Then the mapping has a unique fixed point and therefore the equation $v = Sv$ has a solution over $0 \leq t \leq T$.

We prove now that the solution of (11a, b) is unique; at this time we have uniqueness only with respect to the condition $||v|| \leq M$. Suppose that $v^*(t)$ and $s^*(t)$ is another solution of (11a, b) on $0 \leq t \leq T^*$. We want to show that $v^*(t)$ and $s^*(t)$ coincide with $v(t)$ and $s(t)$ over their common interval of definition, and therefore we may suppose that $T^* \leq T$. Let t_1 be the maximum in $0 \leq t \leq T^*$ of those values of t for which $v(t)$ and $s(t)$ coincide with $v^*(t)$ and $s^*(t)$. We may now write an equation analogous to (11a) with the following changes: The integrations with respect to τ extend from t_1 to t, the integration with respect to ξ extends from 0 to $s(t_1)$, and $\phi'(\xi)$ is replaced by $u_x(\xi, t_1)$, where $u(x, t_1)$ is given by (9)* with $t = t_1$. We write this equation $v = S_1 v$ and note that this equation, together with (11b), has the two distinct solutions $v(t)$, $s(t)$ and $v^*(t)$, $s^*(t)$ on the interval $t_1 \leq t \leq T^*$. Let M^* be the maximum of $v^*(t)$ on this interval. Then $M^* > M$ since the equation $v = Sv$ has a unique solution on $0 \leq t \leq T$ for all v satisfying $||v|| \leq M$. We now select ζ so that following inequalities (30) are all satisfied; here $||q||$ and $||f||$ are the maxima of $q(t)$ and $f(t)$ on $t_1 \leq t \leq t_1 + \zeta$, in $A(T)$ we replace $\phi'(\xi)$ by $u_x(\xi, t_1)$, $\phi'(a)$ by $u_x(s(t_1), t_1)$, and the limits in the integral are 0 and $s(\tau_1)$. Similar remarks apply to the $B_k(T)$, i.e., we replace a by $s(t_1)$, and in $F(M, T)$ we replace $\phi'(x)$ by $u_x(x, t_1)$.

$$\begin{aligned}
&\text{(a)} \quad 2\alpha(||q|| + kM^*)\zeta \leq s(t_1), \\
&\text{(b)} \quad M^*\alpha(||q|| + kM^*)(\zeta/\pi)^{1/2} + M^*B_1(\zeta) + 2k^{-1}\,||f||\,B_2(\zeta) + A(\zeta) \leq M^* - M, \\
&\text{(c)} \quad k\alpha^2(||q|| + kM^*)M^*\zeta \leq 1, \qquad\qquad (30) \\
&\text{(d)} \quad F(M^*, \zeta) < 1.
\end{aligned}$$

We define now $C(\zeta)$, the Banach space, with the maximum norm, of continuous functions on $t_1 \leq t \leq t_1 + \zeta$, the closed subset $C(\zeta, M^*)$ of functions with norm $\leq M^*$, and the mapping $w = S_1 v$. Then $||w|| \leq$ the left side of (30b) plus $|u_x(s(t_1), t_1)|$, and since this last term is $\leq M$ we have $||w|| < M^*$. Thus $w = S_1 v$ is a mapping of $C(\zeta, M^*)$ into itself and the additional conditions (30c, d) make it a contracting mapping. There is, therefore, a unique fixed point subject to the condition $||v|| \leq M^*$. Since both v and v^* satisfy this condition and are both fixed points we must have $v = v^*$ over $t_1 \leq t \leq t_1 + \zeta$. This implies that t_1 coincides with T^*. We have, therefore, proved the uniqueness of the solution $v(t)$ on $0 \leq t \leq T$.

4. Existence and uniqueness of solution of (11a, b) **for all** $t \leq \sigma$. We prove first two lemmas.

LEMMA 1. If $u(x, t)$, $s(t)$ is a solution of (4) in $0 < x < s(t)$, $0 < t < T$ then $u(x, t) \leq 0$.

Suppose $u(x, t)$ is positive at some point x_1, t_1. Then, since it is nonpositive on $s(t)$ and on $t = 0$, $u(x, t)$ must be positive somewhere on the segment $x = 0$, $0 \leq t \leq t_1$ and achieve its maximum there, say at τ, with respect to its values in $0 \leq x \leq s(t)$, $0 \leq t \leq t_1$. But then $u_x(0, \tau) \leq 0$. Now if $f(t)$ were strictly positive we would have a contradiction and therefore $u(x, t)$ could not be positive. But since $u(x, t)$ depends continuously on the initial and boundary data, and in particular on $f(t)$, $u(x, t)$ cannot be positive even when $f(t)$ is nonnegative.

LEMMA 2. If $v(t)$ and $s(t) > 0$ is a solution of (11a, b) over $0 \leq t < T$ then $v(t)$ and $s(t)$ have finite limits as $t \to T$.

We prove first that $v(t)$ is bounded on $0 \leq t < T$. Since $u(s(t), t) = 0$, by Lemma 1 $v(t) = u_x(s(t), t) \geq 0$. We seek now an upper bound for $v(t)$. Since (4) and (11a, b) are equivalent we may write, for $T - \mu \leq t < T$,

$$v(t) = 2 \int_{T-\mu}^{t} v(\tau) N_x(s(t), t; s(\tau), \tau)\, d\tau$$
$$- 2k^{-1} \int_{T-\mu}^{t} f(\tau) N_x(s(t), t; 0, \tau)\, d\tau$$
$$+ 2 \int_{0}^{s(T-\mu)} u_x(\xi, T - \mu) G(s(t), t; \xi, T - \mu)\, d\xi, \tag{31}$$

where μ is positive and less than T and is to be appropriately chosen below. Let $||v(t, \mu)||$, $||f||$, $||q||$, and $||u_x(\mu)||$ be respectively the maximum on $T - \mu \leq \tau \leq t$ of $v(\tau)$, on $0 \leq t \leq T$ of $f(t)$ and $q(t)$, and on $0 \leq \xi \leq s(T - \mu)$ of $u_x(\xi, T - \mu)$. Since the second term in N_x (that involving $s(t) + s(\tau)$) is negative the first term on the right of (31) is less than

$$\int_{T-\mu}^{t} v(\tau)[-(s(t) - s(\tau))/(t - \tau)] K(s(t), t; s(\tau), \tau)\, d\tau$$
$$= \int_{T-\mu}^{t} v(\tau)\left[\left(\alpha \int_{\tau}^{t} q(\zeta)\, d\zeta - k\alpha \int_{\tau}^{t} v(\zeta)\, d\zeta\right) \Big/ (t - \tau)\right] K(s(t), t; s(\tau), \tau)\, d\tau$$
$$\leq \alpha\, ||q||\, ||v(t, \mu)|| \int_{T-\mu}^{t} [4\pi(t - \tau)]^{-1/2}\, d\tau < \alpha(\mu/\pi)^{1/2}\, ||v(t, \mu)||.$$

The second term on the right of (31) is equal to

$$2k^{-1} \int_{T-\mu}^{t} f(\tau)[s(t)/(t - \tau)] K(s(t), t; 0, \tau)\, d\tau$$
$$\leq 2k^{-1}\, ||f|| \int_{T-\mu}^{t} [s(t)/(t - \tau)] K(s(t), t; 0, \tau)\, d\tau. \tag{32}$$

Introducing $\xi = s^2(t)/4(t - \tau)$ we see that the right side of (32) is less than

$$2\pi^{-1/2} k^{-1}\, ||f|| \int_{0}^{\infty} \xi^{-1/2} e^{-\xi}\, d\xi = k^{-1}\, ||f||.$$

The third term on the right of (31) is $\leq 4\, ||u_x(\mu)||$. Thus

$$v(t) < \alpha(\mu/\pi)^{1/2}\, ||q||\, ||v(t, \mu)|| + k^{-1}\, ||f|| + 4\, ||u_x(\mu)||,$$

and it follows that

$$||v(t, \mu)|| < \alpha(\mu/\pi)^{1/2}\, ||q||\, ||v(t, \mu)|| + k^{-1}\, ||f|| + 4\, ||u_x(\mu)||,$$

or

$$[1 - \alpha(\mu/\pi)^{1/2}\, ||q||]\, ||v(t, \mu)|| < k^{-1}\, ||f|| + 4\, ||u_x(\mu)||. \tag{33}$$

If we choose μ so that $1 - \alpha(\mu/\pi)^{1/2}\, ||q|| > 0$ then from (33) $||v(t, \mu)||$, and therefore $v(t)$, is bounded as $t \to T$.

From (11b) it is clear that $s(t)$ has a finite limit as $t \to T$. In (11a) the second and third integrals on the right have finite limits as $t \to T$. Regarding the first integral if we consider the term of N_x involving $s(t) + s(\tau)$ then the integrand consisting of that term multiplied by $v(\tau)$ is bounded and negative and therefore its integral from 0 to t has a finite limit as $t \to T$. Thus it remains to consider

$$\int_0^t v(\tau)[(s(t) - s(\tau))/(t - \tau)]K(s(t), t; s(\tau), \tau)\, d\tau$$
$$= -\alpha \int_0^t v(\tau)\left[\int_\tau^t q(\zeta)\, d\zeta/(t - \tau)\right]K(s(t), t; s(\tau), \tau)\, d\tau$$
$$+ k\alpha \int_0^t v(\tau)\left[\int_\tau^t v(\zeta)\, d\zeta/(t - \tau)\right]K(s(t), t; s(\tau), \tau)\, d\tau. \tag{34}$$

Since both integrands on the right are nonnegative and bounded both integrals have finite limits as $t \to T$ and therefore so does the integral on the left. Thus the left side of (11a) has a finite limit as $t \to T$ and Lemma 2 is proved.

We can now complete the proof of the theorem. Let t^* be the supremum of the T such that (11a, b) has a solution $v(t)$, $s(t)$ on $0 \le t \le T$ with $s(t) > 0$. It can be proved, as in Sec. 3, that any two solutions coincide over their common interval of definition. Then there is a unique solution of (11a, b) on $0 \le t \le t^*$. In Lemma 2 it was shown that $v(t)$ and $s(t)$ have finite limits as $t \to t^*$. If $s(t^*) > 0$ we may define $u(x, t^*)$ by (9)* and we may, by the local theorem proved in Sec. 3, extend the solution $v(t)$, $s(t)$ past t^*. Thus $s(t^*) = 0$. If in Eq. (6) $t < t^*$ then, since $u(x, t) \le 0$ by Lemma 1, the sum of the first three terms on the right of (6) is positive; if $t = t^*$ then that sum is 0. Thus we see that $t^* = \sigma$. This completes the proof of the theorem.

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