A NOTE ON SINGULARITIES IN A COSSERAT CONTINUUM*

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Abstract. This paper is concerned with the singularities that are due to concentrated couples in an infinite linear, elastic, isotropic Cosserat continuum. The solution to the problem of a concentrated couple, acting within an infinite region, may be obtained as a limiting case of the solution to the problem of body moments acting within a finite portion of the infinite medium. Alternatively, the solution to the problem of the concentrated couple can be constructed from the solution to the case of a concentrated force acting within an infinite body, by combining two double-forces with moments to form a center of rotation.

In this paper it is shown that in a Cosserat continuum the two above mentioned singular solutions to the case of a concentrated couple, acting within an infinite body, are not the same. By means of a specific linear combination of these two singular solutions it is possible to reconstruct the classical center of rotation, which is accompanied by an additional micro-rotation field. It is shown that there exists a limiting case in which the macro-displacements are eliminated altogether, resulting in a singular field of micro-rotations alone.

1. The equations of a linear, elastic, isotropic Cosserat continuum. In a Cosserat continuum [1]**, deformations are characterized by two kinematical variables: the displacement \( u_i \) and the independent, rigid, anti-symmetric micro-rotation \( \psi_{i[i]} \). The quantities \( \psi_{i[i]} \) describe a rigid rotation of some material “superstructural” property (e.g. the Cosserat triad or, alternatively, a “micro-structure”).

Following Mindlin’s formulation [2] of the linear, elastic case we define

\[
\begin{align*}
\epsilon_{ii} &= \frac{1}{2}(u_{i,i} + u_{i,i}), \\
\gamma_{i[i]} &= \frac{1}{2}(u_{i,i} - u_{i,i}) - \psi_{i[i]}, \\
\kappa_{i[ik]} &= \psi_{i[ik]}. & (1)
\end{align*}
\]

Then, for an isotropic, centrosymmetric medium the constitutive relations are

\[
\begin{align*}
\tau_{ij} &= \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}, \\
\sigma_{i[i]} &= 2\beta \gamma_{i[i]}, \\
\mu_{i[ik]} &= \alpha_i (k_{i[i]} \delta_{ik} + k_{i[k]} \delta_{ik}) + 2\alpha_2 \kappa_{i[ik]} + \alpha_3 (k_{i[i]} + k_{i[k]}),
\end{align*}
\]

where \( \tau_{ij} \) is the classical “Cauchy” stress, \( \sigma_{i[i]} \) is the anti-symmetric part of Mindlin’s relative stress, and \( \mu_{i[ik]} \) is the Cosserat’s couple-stress. The quantities \( \alpha_i \) and \( \beta \) are

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**Numbers in square brackets indicate the reference listed at the end of this paper.
some material constants. The dimension of $\alpha_i$ differs from the dimension of the Lamé constants $\lambda, \mu$ by the square of length.

The equations of equilibrium read:

$$
(t_{ij} + \sigma_{ij})_{,i} + f_i = 0,
$$

$$
\mu_{\{ij\},i} + \sigma_{\{ij\}} + \phi_{\{ij\}} = 0.
$$

In (3) $f_i$ and $\phi_{\{ij\}}$ denote body force and body couple, respectively.

The boundary conditions, at a boundary with outward normal $n_i$, are

$$
t_i = n_i(t_{ij} + \sigma_{ij}),
$$

$$
T_{\{ijk\}} = n_i\mu_{\{ijk\}}.
$$

Substituting (1) in (2), and then in (3) we obtain the kinematical equations of motion

$$(\lambda + \mu - \beta)u_{\{ij\}} + (\mu + \beta)u_{\{ij\}} - 2\beta\psi_{\{ij\},i} + f_i = 0,$$

$$(\alpha_1 + \alpha_3)(\psi_{\{ij\},kj} + \psi_{\{jk\},ki}) + 2\alpha_2\psi_{\{ij\},kk} - 2\beta\psi_{\{ij\}} + \beta(u_{\{ij\},i} - u_{\{i,j\}}) + \phi_{\{ij\}} = 0.$$

Employing the “direct” notation, Eqs. (5) read

$$(\lambda + \mu - \beta)\nabla \cdot \mathbf{u} + (\mu + \beta)\nabla^2 \mathbf{u} - 2\beta\nabla \cdot \mathbf{\psi}^A + \mathbf{f} = 0,$$

$$(\alpha_1 + \alpha_3)(\nabla \cdot \mathbf{\psi}^A\nabla + \nabla \cdot \mathbf{\psi}^A\nabla) + 2\alpha_2\nabla^2 \mathbf{\psi}^A - 2\beta \mathbf{\psi}^A + \beta(\nabla \mathbf{u} - \mathbf{u} \nabla) - \frac{1}{3} \mathbf{I} \times \mathbf{c} = 0.$$

In (6) the quantities $\mathbf{u}$ and $\mathbf{f}$ are vectors with components $u_i$ and $f_i$, and $\mathbf{\psi}^A$, $\mathbf{\phi}^A$ and $\mathbf{I}$ are dyadics with components $\psi_{\{ij\}}$, $\phi_{\{ij\}}$ and $\delta_{ij}$. The quantity $\phi^A$ has been written in terms of a body couple vector $\mathbf{c}$,

$$
\phi^A = -\frac{1}{3} \mathbf{I} \times \mathbf{c}.
$$

Mindlin has shown [2], that a complete solution of (6), or (5), can be expressed as

$$
\mathbf{u} = \nabla \times \mathbf{K} + (1 - l_2^2 \nabla^2)(\mathbf{B} - l_2^2 \nabla \times \mathbf{B}) - \frac{1}{3}(k_1 - l_2^2 \nabla^2)\nabla [r \cdot (1 - l_2^2 \nabla^2)\mathbf{B} + \mathbf{B}_0],
$$

$$
\mathbf{\psi}^A = -\frac{1}{3} \mathbf{I} \times [\nabla^2 \nabla (r \cdot \mathbf{K} + \mathbf{K}_0) + 2 \nabla \times \mathbf{B}]
$$

where $\mathbf{B}$, $\mathbf{B}_0$, $\mathbf{K}$, and $\mathbf{K}_0$ are stress-functions of the Boussinesq–Papkovitch type. These functions satisfy the following relations:

$$
\mu(1 - l_2^2 \nabla^2)\nabla^2 \mathbf{B} = -\mathbf{f} - \frac{\mu + \beta}{2\beta} \nabla \times \mathbf{c},
$$

$$
\mu \nabla^2 \mathbf{B}_0 = r \left[ \mathbf{f} + \frac{\mu + \beta}{2\beta} \nabla \times \mathbf{c} \right],
$$

$$
2\beta \nabla^2 \mathbf{K} = \mathbf{c},
$$

$$
2\beta(1 - l_2^2 \nabla^2)\nabla^2 \mathbf{K}_0 = 4l_2^2 \nabla \cdot \mathbf{c} - r \cdot (1 - l_2^2 \nabla^2)\mathbf{c}
$$

where, in (7) and (8)

$$
k_1 = \frac{\lambda + \mu}{\lambda + 2\mu},
$$

$$
l_1^2 = (2\alpha_2 - \alpha_1 - \alpha_3) \frac{\mu + \beta}{2\mu\beta},
$$

$$
l_2^2 = \frac{\alpha_2}{\beta}, \quad l_3^2 = \frac{2\alpha_2 - \alpha_1 - \alpha_3}{2\beta},$$

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2. The singular solutions for concentrated force and couple. Mindlin [2] has given the solution to the problems of a concentrated force and a concentrated couple acting within an infinite medium.

For a concentrated force $P$, acting at the origin of a Cartesian coordinate system $x, y, z$ the stress-functions are

$$B = \frac{P}{4\pi \mu} g_1 ,$$
$$B_0 = 0 ,$$
$$\mathbf{K} = 0 ,$$
$$K_0 = 0 .$$

For a concentrated moment $C$, acting at the origin the stress functions are

$$B = \frac{\mu + \beta}{8\pi \mu \beta} C \times \nabla g_1 ,$$
$$B_0 = 0 ,$$
$$\mathbf{K} = \frac{C}{8\pi \beta r} ,$$
$$K_0 = -\frac{r^2}{4\pi \beta} C \cdot \nabla g_2 .$$

In (9) and (10) $g_i = (1 - e^{-r/\lambda})/r$.

Let $u^{(1)}$ and $\psi^{(1)}$ denote the kinematical field due to a concentrated force $P \mathbf{e}_z$ acting at the origin, and $u^{(2)}$, $\psi^{(2)}$ the kinematical field that is due to a force $P \mathbf{e}_z$ acting at the origin.

Employing the fields $u^{(1)}$, $\psi^{(1)}$ and $u^{(2)}$, $\psi^{(2)}$, it is possible to construct a singular solution due to a "center of rotation about the axis of $z$" [3]. We let the forces $h^{-1} P \mathbf{e}_z$ and $-h^{-1} P \mathbf{e}_z$ act at the origin $(0, 0, 0)$, and the forces $-h^{-1} P \mathbf{e}_z$ and $h^{-1} P \mathbf{e}_z$ act at $(0, h, 0)$ and $(h, 0, 0)$, respectively, as shown in Fig. 1.

Passing to the limit as $h \to 0$ the resulting kinematical field is given by

$$u^{(R)} = \frac{\partial u^{(1)}}{\partial y} - \frac{\partial u^{(2)}}{\partial x} ,$$
$$\psi^{A(R)} = \frac{\partial \psi^{(1)}}{\partial y} - \frac{\partial \psi^{(2)}}{\partial x} .$$

Computing the field $u^{(R)}$ and $\psi^{A(R)}$ we obtain

$$u^{(R)} = -\frac{P}{4\pi \mu} (1 - l_y^2 \nabla^2) \mathbf{e}_z \times \nabla g_1 ,$$
$$\psi^{A(R)} = \frac{P}{8\pi \mu} I \times \nabla \times \mathbf{e}_z \times \nabla g_1 .$$

Expressions (12) may be obtained from (7) if we select
Fig. 1. The system of concentrated forces which yields, in the limit as $h \to 0$, a "center of rotation about the axis of $z$.

\[
B = B^{(R)} = -\frac{P}{4\pi\mu} e_\tau \times \nabla g_1 ,
\]
\[
B_0^{(R)} = K^{(R)} = K_0^{(R)} = 0 .
\] (13)

For a concentrated moment directed about the $z$ axis, at the origin $C = Ce_\tau$, Eqs. (10) yield

\[
B^{(C)} = -\frac{\mu + \beta}{8\pi\mu\beta} Ce_\tau \times \nabla g_1 = \frac{\mu + \beta}{2\beta} \frac{C}{P} B^{(R)} ,
\]
\[
B_0^{(C)} = 0 ,
\]
\[
K^{(C)} = -\frac{C}{8\pi\beta r} e_\tau ,
\]
\[
K_0^{(C)} = -\frac{r^2 C}{4\pi\beta} \frac{\partial g_1}{\partial z} .
\] (14)

A comparison between (13) and (14) shows that the singularities due to the two kinds of concentrated couples are not the same. They differ by the stress functions $K$ and $K_0$, given in (14). These functions yield a self-equilibrating kinematical field.

It may be worth noting that in the case of couple-stress theory [4], the singularity due to a concentrated couple and the singularity due to a "center of rotation" are the same.

3. Special cases. (a) Consider the following linear combination of the solutions to a concentrated couple $S^{(C)}$ and a center of rotation $S^{(R)}$.

\[
S^{(L)} = -\frac{2P}{C} \frac{\beta}{\mu} S^{(C)} + \frac{\mu + \beta}{\mu} S^{(R)} .
\] (15)
Then

\[ B^{(L)} = -\frac{2P}{C} \beta B^{(c)} + \frac{\mu + \beta}{\mu} B^{(R)} = 0, \]

\[ B_0^{(L)} = 0, \tag{16} \]

\[ K^{(L)} = -\frac{2P}{C} \beta K^{(c)} = \frac{P}{4\pi \mu} e_z/r, \]

\[ K_0^{(L)} = -\frac{2P}{C} \beta K_0^{(c)} = \frac{P\delta_{g_z}}{2\pi \mu \partial}. \]

The corresponding kinematical field is

\[ u^{(L)} = \frac{P}{4\pi \mu} \nabla \times \frac{e_z}{r}, \tag{17} \]

\[ \psi^{A(L)} = \frac{P}{8\pi \mu} I \times \nabla \frac{\partial g_z}{\partial z}. \]

It is interesting to note that the expression for \( u^{(L)} \) has thus been made to agree with the classical result for a center of rotation about the axis of \( z \). It does not depend on the “micro-parameters” of the Cosserat medium.

(b) Consider the limit of the solution \( S^{(c)} \), for a concentrated moment about the \( z \) axis, as the ratio \( \mu/\beta \to \infty \).

In this case the characteristic lengths \( l_1^2 \to l_3^2 \) and

\[ (1 - l_3^2 \nabla^2) g_1 \to 1/r. \]

The kinematical fields become

\[ u^{(c)} \to 0, \tag{18} \]

\[ \psi^{A(c)} = -\frac{C}{10\pi \beta} I \times \left[ \frac{e^{-r/l_1}}{r l_1} e_z + \nabla \left( \frac{\partial g_1}{\partial z} - \frac{\partial g_2}{\partial z} \right) \right]. \]

It is seen that for this limiting case the macro-displacements vanish, and the resulting singular field contains micro-rotations alone.

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