A SOLUTION OF VAN DER POL'S DIFFERENTIAL EQUATION*

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The object of the present note is to form an elliptic function of the second order:

\[ x = \varphi(t | 2\omega_1 , 2\omega_3) , \]

\( \omega_1 \) being real and \( \omega_3 \) purely imaginary, which satisfies the well-known differential equation of Van der Pol:

\[ x'' + \mu(x^2 - 1)x' + \kappa^2 x = 0, \quad (\prime = d/dt) \]

under the initial condition that \( x = x_0 \) when \( t = t_0 \).

Substituting in (1) the following relations with the usual notations [1] in the theory of elliptic functions:

\[ x = \varphi(t | 2\omega_1 , 2\omega_3), \quad (\varphi')^2 = 4\varphi^3 - g_2 \varphi - g_3, \quad \varphi'' = 6\varphi^2 - \frac{1}{2}g_2, \quad (\prime = d/dt) \]

we get

\[ 6\varphi^2 - \frac{1}{2}g_2 + \mu(\varphi^2 - 1)(4\varphi^3 - g_2 \varphi - g_3)^{1/2} + \kappa^2 \varphi = 0, \]

or, by clearing off the radical,

\[ 4\mu^2 \varphi^7 - (8 + g_2)\mu^2 \varphi^5 - (36 + \mu^2 g_3)\varphi^3 + \{(4 + 2g_3)\mu^2 - 12\kappa^2\}\varphi^3 \\
+ (6g_2 + 2\mu^2 g_3 - \kappa^4)\varphi^3 - (\mu^2 - \kappa^2)g_2 \varphi - \mu^2 g_3 - \frac{1}{4}g_2^2 = 0. \quad (2) \]

Next, for the purpose of settling the fundamental periods \( 2\omega_1 , 2\omega_3 \), let us take

\[ -\mu^2 g_3 = \frac{1}{4}g_2^2, \]

then from (2) and (3), we obtain

\[ 4\mu^2 \varphi^7 - (8 + g_2)\mu^2 \varphi^5 - (36 - \frac{1}{4}g_2^2)\varphi^3 + \{(4 + 2g_3)\mu^2 - 12\kappa^2\}\varphi^3 \\
- (\kappa^4 - 6g_2 + \frac{1}{2}g_2^2)\varphi - (\mu^2 - \kappa^2)g_2 = 0, \quad (4) \]

or, by the substitution of the initial condition,

\[ x_0(x_0^2 - 2)g_2 + 24[x_0 + \frac{1}{2}\kappa^2 - \frac{1}{8}\mu^2(x_0^2 - 1)]g_2 \\
- 144[x_0^3 + \frac{1}{4}\kappa^2 x_0^2 + \frac{3}{8}\kappa x_0 - \frac{1}{8}\mu^2 x_0(x_0^2 - 1)] = 0. \quad (5) \]

In the present case, both \( g_2 \) and \( g_3 \) must be real; besides, \( g_3 \) must also be negative. Now, we have the expressions:

\[ g_2 = \left( \frac{\pi}{\omega_1} \right)^4 \left\{ \frac{1}{12} + 20 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{2n}} \right\}, \]

\[ -g_3 = \left( \frac{\pi}{\omega_1} \right)^6 \left\{ \frac{7}{3} \sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1 - q^{2n}} - \frac{1}{216} \right\}, \quad q = \exp (\omega_3 \pi i/\omega_1), \quad i = (-1)^{1/2} \]

where \( q \) will be later seen a positive quantity less than 1.

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If \( q \) is approximately greater than \((505)^{-1/2}\), then \(-g_3 > 0\) and (3) holds. If
\[
\left\{ x_0 + \frac{4}{3} x^2 - \frac{1}{2} \mu^2 (x_0^2 - 1)^2 \right\}^2 + x_0 (x_0^2 - 2) \left\{ x_0^2 + \frac{1}{2} x^2 + \frac{1}{2} k^2 x_0 - \frac{1}{2} \mu^2 (x_0^2 - 1)^2 \right\} \geq 0,
\]
we see by (5) that \( g_2 \) is real. In the case that both \( \mu \) and \( k \) are comparatively smaller than \( x_0 \), we see that the condition is satisfied, for
\[
x_0^2 + x_0^2 (x_0^2 - 2) = x_0^2 (x_0^2 - 1)^2 \geq 0.
\]

Under the above restrictions, we can thus calculate \( \omega_1 \) and \( \omega_3 \) with the values of \( g_2 \) and \( g_3 \).

By solving the three equations:
\[
g_1 = -4(e_1 e_2 + e_2 e_3 + e_3 e_4),
g_3 = 4e_1 e_2 e_3,\]
\[
0 = e_1 + e_2 + e_3
\]
we obtain a set of real quantities \( e_1, e_2, e_3 \) such that \( e_1 > e_2 > e_3 \), if \( 0 < g_2 < (16/27) \mu^4 \); consequently
\[
k^2 = (e_2 - e_3)/(e_1 - e_3), \quad k'^2 = (e_1 - e_2)/(e_1 - e_3).
\]

Finally, we can compute \( K, K', \omega_1, \omega_3 \) by the formulae:
\[
K = (\pi/2) F(\frac{1}{2}, \frac{1}{2}; 1; k^2), \quad K' = (\pi/2) F(\frac{1}{2}, \frac{1}{2}; 1; k'^2),
\]
\[
\omega_1 = K(e_1 - e_3)^{-1/2}, \quad \omega_3 = iK'(e_1 - e_3)^{-1/2}.
\]
\[
q = \exp(\omega_3 i/\omega_1) = \exp(-K' \pi / K), \text{ which is real and less than 1.}
\]

Various properties of the solutions of (1) usually discussed can be derived directly from the explicit expression of the solution we have obtained, namely
\[
x(t) = \varphi(t | 2\omega_1, 2\omega_3)
\]
\[
= -\frac{\eta_1}{\omega_1} + \left( \frac{\pi}{2\omega_1} \right)^2 \csc^2 \pi v - 2 \left( \frac{\pi}{\omega_1} \right)^2 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} \cos 2n \pi v
\]
where
\[
v = \frac{t}{2\omega_1}, \quad \eta_1 = \zeta(\omega_1) = \frac{\pi^2}{\omega_1} \left\{ \frac{1}{12} - 2 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} \right\}.
\]

Consider, for example, the path on the phase plane. Now
\[
y = x' = \varphi'(t | 2\omega_1, 2\omega_3), \quad (') = d/dt
\]
and the equation of the path runs as follows,
\[
y^2 = 4x^3 - g_2 x + (1/4 \mu^2) g_2^2 = 4(x - e_1)(x - e_2)(x - e_3).
\]

Since
\[
e_1 + e_2 + e_3 = 0,
e_1 > e_2 > e_3,
e_1 e_2 e_3 = -(1/4 \mu^2) g_2^2 < 0,
\]
we find easily

\[ e_3 < 0, \quad e_1 > e_2 > 0; \]

the ordinates are then imaginary for \( x < e_3, e_2 < x < e_1 \). Hence we see that there exists always, and evidently only, one cycle on the phase plane for each solution under consideration as shown in the figure.

Reference

[1] F. Oberhettinger und W. Magnus, *Anwendung der Elliptischen Funktionen in Physik und Technik*, Springer-Verlag, Berlin, 1949; first chapter (where \( \omega = \omega_1, \omega' = \omega_2 \)).