SMALL FINITE DEFORMATIONS OF ELASTIC DIELECTRICS*

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1. Introduction. As in finite elasticity theory, there are certain problems in the theory of finite deformations of elastic dielectrics which can be solved exactly by the inverse method. The deformation and the electric field are prescribed at the outset and it is then verified that the resulting state can be supported without mechanical body force or distributed charge in every homogeneous, isotropic, incompressible, elastic dielectric. Singh and Pipkin [1] have described all such possible combinations and call them controllable states of these materials. Since these states do not require any detailed knowledge of the form of the stored-energy function for the material, they could be used in an experimental determination of the stored-energy function. While there is a moderately large number of controllable states for incompressible, dielectrics [1], Singh [2] has proved that if the dielectric considered is compressible, then the only controllable states possible are the combinations of pure homogeneous deformations and the uniform electric fields. For a problem in which the displacement or the electric field, prescribed initially, are other than those in controllable states, progress towards the solution is sometimes possible if certain specific forms of the stored-energy function are made available.

In finite elasticity theory, the development of small but finite deformations was consistently developed by Murnaghan [3] and later by Rivlin [4]. In the present paper we carry out a similar investigation for elastic dielectrics. The stored-energy is assumed to be a function of the deformation gradients and the total electric field present. Constitutive relations relating stress, strain, electric field, and the dielectric displacement field are suggested. The special polynomial forms of the stored-energy function of an initially homogeneous, isotropic, elastic dielectric are developed for small finite deformations and weak electric fields. The application of the corresponding approximate constitutive equations is then illustrated by obtaining the solution of a problem which cannot be solved on the basis of a completely arbitrary stored-energy function.

Certain polynomial approximations have also been considered by Toupin [5], and Eringen [6]. No analytical solutions based on such approximations have been obtained as far as we know. The approximations arrived at in this paper are different from those in [5] and [6]. The analysis in this presentation assumes the total electric field rather than polarization as the independent variable. Also, another difference is that we do not decompose the stress, the electric field, or the dielectric displacement field into sums of various parts. When the electrical effects are not present, it is shown that the approximate forms of the stored-energy function reduce to the well-known strain-energy forms of Murnaghan [3], Mooney–Rivlin materials [7], and Neo–Hookean solids [8], in the theory of finite deformations of elastic solids.

In Secs. 2 and 3 the basic equations of the theory of elastic dielectrics are restated

from [1]. This theory we believe is, in essence, equivalent to that formulated by Toupin [5], [9], where the discussion of fundamental aspects is also given.

Following the scheme suggested by Rivlin [10] in finite elasticity theory, we derive the first order and second order approximate polynomial forms of the stored-energy function in Secs. 4–6. These special forms, although valid only for small strains and weak electric fields, nevertheless admit arbitrary rotations. These forms are further specialized to cover the case of classical electrostriction when rotations are also small.

In the remainder of the paper, we examine the problem of simultaneous extension—a simple shear parallel to the axis and a simple shear about the axis—of a long circular tube of a homogeneous, isotropic, incompressible, elastic dielectric material in the presence of a radial electric field. Singh and Pipkin [1] have shown that if the functional dependence of the stored-energy function is arbitrary, the deformation and the field described above cannot together be supported without mechanical body force or charge distribution. However, it is demonstrated in this presentation that if attention is confined to small finite deformations and weak electric fields, then the problem can be solved. Closed form expressions for shear functions and the surface tractions necessary to support the prescribed combination of deformation and the electric field are determined within the first order approximation.

Finally, we may point out that with the procedure developed, because of its simplicity, solutions to many other problems can be obtained, but have not been included here since the method of treatment is now made clear.

We use the familiar suffix notation, summation convention, and Cartesian coordinate systems unless specified otherwise.

2. Continuum electrostatics. We reproduce here those basic equations of continuum electrostatics and mechanics which are independent of the composition of the material media that may be involved.

The macroscopic electric field E_i is conservative:

$$E_{i,j} = E_{j,i} . \tag{2.1}$$

In the assumed absence of distributed charge, the dielectric displacement field D_i is solenoidal:

$$D_{i,i} = 0. \tag{2.2}$$

At the charge-free surface of a dielectric with outward unit normal n_i, the tangential component of E_i and the normal component of D_i are continuous:

$$e_{ijk}(E_j^{(0)} - E_j)n_k = 0, \qquad (D_i^{(0)} - D_i)n_i = 0, \tag{2.3}$$

where $E_i^{(0)}$ and $D_i^{(0)}$ are evaluated in the outside medium, and E_i and D_i are evaluated inside the dielectric.

We assume that the resultant force F_i and moment G_i exerted on the material contained in an arbitrary volume V, not including gravitational or inertial forces and moments, can be expressed entirely in terms of a stress vector t_i acting over the surface S of the volume V, in the forms:

$$F_i = \oiint_S t_i \, dS, \qquad G_i = \oiint_S e_{ijk}x_j t_k \, dS. \tag{2.4}$$

Here x_i are Cartesian coordinates, and G_i is the moment about the origin. The stress

vector t_i accounts for all electro-mechanical effects except the gravitational and mechanical body forces which will be set equal to zero in the work to follow. We specifically exclude surface couples and body couples.

On the basis of the assumptions (2.4), one obtains as in continuum mechanics,

$$t_i = \sigma_{ij} n_j \quad \text{on the surface } S, \tag{2.5}$$

that the stress matrix is symmetric,

$$\sigma_{ij} = \sigma_{ji} , \tag{2.6}$$

and that the differential form of the equation of translational equilibrium, when gravitational or inertial forces are neglected, is

$$\sigma_{ij,i} = 0. \tag{2.7}$$

If T_i represents the mechanical force per unit area applied on the boundary of the dielectric body, then

$$T_i = (\sigma_{ij} - \sigma_{ij}^{(0)})n_j , \tag{2.8}$$

where the superscript zero carries the same meaning as in (2.3). In the absence of electrical effects, at the surface of a body, one ordinarily takes the stress $\sigma_{ij}^{(0)}$ in the surrounding free space to be zero. In the present case, however, there will be a nonzero Maxwell stress everywhere.

3. Constitutive equations.   In free space,

$$D_i = \epsilon E_i , \tag{3.1}$$

ϵ being the physical constant for free space.

The stress $\sigma_{ij}^{(0)}$ in free space is the Maxwell stress defined by

$$\sigma_{ij} = \epsilon E_i E_j - (\epsilon/2) E_k E_k \; \delta_{ij} . \tag{3.2}$$

Henceforth the dielectric bodies that we consider are surrounded by a charge free medium obeying constitutive equations of the forms (3.1) and (3.2). Apparently, the stress (3.2) satisfies (2.7) identically by virtue of (2.1), (2.2), and (3.1).

The deformation of a material dielectric body may be described by specifying coordinates $x_i(X_1 , X_2 , X_3)$, in the deformed state, of the generic particle which initially was located at X_i in the undeformed state, all coordinates being measured with respect to a single fixed rectangular Cartesian system x. The elastic dielectric bodies to which we confine our discussion are those for which the stored-energy W as a function of deformation gradients $\partial x_i / \partial X_j$ and the electric field E_i exists, resulting in the constitutive relations:

$$\sigma_{ij} = \rho \frac{\partial W}{\partial(\partial x_i / \partial X_k)} \frac{\partial x_j}{\partial X_k} + \rho \frac{\partial W}{\partial E_i} E_j , \tag{3.3}$$

and

$$D_i = \partial W / \partial E_i , \tag{3.4}$$

where ρ is the mass density measured in the deformed configuration of the dielectric.

Constitutive relations somewhat equivalent to (3.3) and (3.4) were originally derived by Toupin [5] through a principal of virtual work by postulating the existence of a

stored-energy function. In a latter paper [9], a different analysis was employed yielding the same constitutive equations. The theory suggested here in the form of relations (3.3) and (3.4) differs formally from Toupin's theory in that we take the electric field E_i , rather than the polarization, as the independent variable. No attempt has been made in the present paper at the formulation of an energy principle which should yield the relations (3.3) and (3.4).

Further restrictions upon the form of W can be obtained from the requirement that if the dielectric body is subjected to a rigid rotation, together with the field E_i , then the force system will undergo the same rotation. Under this assumption it follows that the stored-energy function must be expressible in the form

$$W = W((\partial x_k/\partial X_p)(\partial x_k/\partial X_q), (\partial x_p/\partial X_q)E_p). \qquad (3.5)$$

If the medium is holohedral isotropic in its undeformed field free state, then the stored-energy W can be further restricted to be a function of the six scalar invariants [11]:

$$W = W(I_1 , I_2 , I_3 , I_4 , I_5 , I_6), \qquad (3.6)$$

where

$$I_1 = g_{ii} , \qquad I_2 = \tfrac{1}{2}[g_{ii}g_{jj} - g_{ij}g_{ji}], \qquad I_3 = \mathrm{Det}\,|g_{ij}|,$$
$$I_4 = E_i E_i , \qquad I_5 = E_i g_{ij} E_j , \qquad I_6 = E_i g_{ik} g_{kj} E_j . \qquad (3.7)$$

The matrix g_{ij} is the Finger strain tensor defined by

$$g_{ij} = (\partial x_i/\partial X_k)(\partial x_j/\partial X_k). \qquad (3.8)$$

The substitution of W given by (3.6) into the constitutive equations (3.3) and (3.4) gives

$$\sigma_{ij} = \frac{2}{(I_3)^{1/2}} \left[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) g_{ij} - \frac{\partial W}{\partial I_2} g_{ij}^2 + I_3 \frac{\partial W}{\partial I_3} \delta_{ij} \right.$$
$$+ \frac{\partial W}{\partial I_4} E_i E_j + \frac{\partial W}{\partial I_5} (g_{ik}E_i E_k + g_{ik}E_k E_j)$$
$$\left. + \frac{\partial W}{\partial I_6} (g_{ik}^2 E_i E_k + g_{ik}^2 E_j E_k) + \frac{\partial W}{\partial I_6} g_{ik}g_{jl}E_k E_l \right], \qquad (3.9)$$

and

$$D_i = 2\left(\frac{\partial W}{\partial I_4} \delta_{ij} + \frac{\partial W}{\partial I_5} g_{ij} + \frac{\partial W}{\partial I_6} g_{ij}^2 \right) E_j . \qquad (3.10)$$

We use g_{ij}^2 to denote the ij-component of the square of the matrix **g**.

Considering the stress and the dielectric displacement field as functions of the deformation gradients and the electric field, Singh and Pipkin [1] formulated relations equivalent to (3.9) and (3.10) with the difference that they did not use the expressions for response coefficients in terms of derivatives of the stored-energy.

4. Approximate theories. Assuming the stored-energy function W in (3.6) to be a polynomial function of its arguments, we may without loss of generality write

$$W = \sum_{\alpha,\beta,\gamma,\delta,\lambda,\mu} A_{\alpha\beta\gamma\delta\lambda\mu}(I_1 - 3)^\alpha(I_2 - 3)^\beta(I_3 - 1)^\gamma I_4^\delta I_5^\lambda I_6^\mu , \qquad (4.1)$$

where $A_{\alpha\beta\gamma\delta\lambda\mu}$ are constants of the material. It is readily noted from (4.1) that if the

medium is undeformed and field free,

$$I_1 - 3 = I_2 - 3 = I_3 - 1 = I_4 = I_5 = I_6 = 0.$$

Let e_1, e_2, e_3 denote the principal extensions at a point P of the dielectric body and E_1, E_2, E_3 the components of the electric field referred to the principal directions of strain at P. The invariants (3.7) then become

$$I_1 = (1 + e_1)^2 + (1 + e_2)^2 + (1 + e_3)^2, \qquad I_3 = (1 + e_1)^2(1 + e_2)^2(1 + e_3)^2,$$

$$I_2 = (1 + e_1)^2(1 + e_2)^2 + (1 + e_2)^2(1 + e_3)^2 + (1 + e_3)^2(1 + e_1)^2, \qquad I_4 = E_i E_i ,$$

$$I_5 = (1 + e_1)^2 E_1^2 + (1 + e_2)^2 E_2^2 + (1 + e_3)^2 E_3^2 ,$$

$$I_6 = (1 + e_1)^4 E_1^2 + (1 + e_2)^4 E_2^2 + (1 + e_3)^4 E_3^2 . \tag{4.2}$$

If $I_1 - 3, I_2 - 3, I_3 - 1, I_4, I_5$, and I_6, are sufficiently small, we may approximate W, to any desired order in the principal extensions and powers of the electric field components, by neglecting terms above an appropriate degree in the series expansion (4.1). Any such approximate form of W will be invariant to all rigid rotations simultaneously of the dielectric body and the electric field. It is evident, however, that the quality of the approximation sought will depend upon the magnitude of the physical constants $A_{\alpha\beta\gamma\delta\lambda\mu}$.

5. First approximation. Suppose the principal extensions e_i are small and the electric field E_i sufficiently weak. We may define the first approximation as one in which we retain in W all terms involving principal extensions e_i to a lower degree than the third, the terms containing the field components E_i to a degree lower than the fourth, and the product terms only of the type $e_i E_j^2$. In order to obtain the form of W for such an approximation, we will have to retain twenty-two terms in the series expansion (4.1). However, we can arrive at the same approximation involving fewer terms by introducing a new set of six mutually independent scalar invariants:

$$J_1 = I_1 - 3, \qquad J_2 = (I_2 - 3) - 2(I_1 - 3),$$

$$J_3 = (I_3 - 1) - (I_2 - 3) + (I_1 - 3), \tag{5.1}$$

$$J_4 = I_4 , \qquad J_5 = I_5 - I_4 , \qquad J_6 = I_6 - 2(I_5 - I_4) - I_4 .$$

In place of (4.1) we now have

$$W = \sum_{\alpha,\beta,\gamma,\delta,\lambda,\mu} B_{\alpha\beta\gamma\delta\lambda\mu} J_1^\alpha J_2^\beta J_3^\gamma J_4^\delta J_5^\lambda J_6^\mu , \tag{5.2}$$

where B's are physical constants of the material.

In view of (4.2), (5.1), and the statement of the first approximation, the following form for the stored energy function is obtained from (5.2):

$$W = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 . \tag{5.3}$$

To facilitate the writing, a's are being used here to denote the B's in (5.2). Since we may, without loss of generality, assume vanishing stored-energy and stress in the initial undeformed field free state, Eqs. (5.3) and (3.9) yield $a_0 = a_1 = 0$, so that (5.3) reduces to

$$W = a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 . \tag{5.4}$$

We may remark here that when electrical effects are absent, the last three terms

in (5.4) drop out leaving behind the first approximate form of the strain-energy function in the theory of finite elasticity [10].

Introduction of (5.4) into (3.9) and (3.10) results in the following constitutive relations:

$$\sigma_{ij} = 2\rho[[a_2 + (a_2 + 2a_3)J_1 + a_6 J_4]g_{ij}$$

$$- a_2 g_{ij}^2 + (a_4 - a_5)E_i E_j + a_5(g_{ik}E_k E_j + g_{jk}E_k E_i)], \qquad (5.5)$$

and

$$D_i = 2[(a_4 - a_5)\, \delta_{ij} + a_5 g_{ij}]E_j . \qquad (5.6)$$

In the expression (5.4) for W if we further neglect terms of degree higher than the second in displacement gradients $\partial u_i/\partial X_j$ and the field components E_i , and product terms of order higher than $(\partial u_i/\partial X_j)E_k^2$, then we obtain

$$W = 2a_2(e_{ij}e_{ij} - e_{ii}e_{jj}) + 4a_3 e_{ii}e_{jj} + a_4 E_i E_i + 2a_5 e_{ij}E_i E_j + 2a_6 e_{ii}E_k E_k , \qquad (5.7)$$

where e_{ij} is the classical linear strain tensor defined by

$$e_{ij} = \tfrac{1}{2}(\partial u_i/\partial X_j + \partial u_j/\partial X_i), \qquad (5.8)$$

with

$$u_i = x_i - X_i . \qquad (5.9)$$

One may then designate W given by (5.7) as the stored-energy function for the classical coupled theory of electrostriction. Both the approximate forms (5.4) and (5.7) are valid only for small strains and weak electric fields. However, it is essential to note the distinction between these two types of approximations. Whereas the form (5.4) admits arbitrary rotations, only infinitesimal rotations, even rigid, are permissible in the application of (5.7).

6. Second approximation. A second approximation can be arrived at by retaining in the expansion (5.2) for W terms up to and including third powers in principal extensions e_i and field components E_i , and product terms up to and including $e_i^2 E_j^2$. To this end the corresponding expression for W is

$$W = a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 + a_7 J_1^3$$

$$+ a_8 J_1 J_2 + a_9 J_3 + a_{10}J_2 J_4 + a_{11}J_1^2 J_4 + a_{12}J_6 + a_{13}J_1 J_5 , \qquad (6.1)$$

where a's, as before, are material constants. In writing (6.1), it is assumed that W and stresses vanish in the undeformed field free state.

We note that when the electric field is not present, then (6.1) reduces to the well-known Murnaghan's form [3] in the theory of finite deformations of elastic solids.

The corresponding approximate form for W in the classical sense when rotations are also small can now be obtained by further neglecting in (6.1) those terms which are of a degree higher than the third in the quantities $\partial u_i/\partial X_j$ or E_i as well as product terms of order higher than $(\partial u_i/\partial X_j)^2 E_k^2$. This approximation yields

$$W = 2(a_2 + 2a_3)\left(\frac{\partial u_i}{\partial X_i}\right)^2 - a_2\left(\frac{\partial u_i}{\partial X_k}\frac{\partial u_i}{\partial X_k} + \frac{\partial u_i}{\partial X_k}\frac{\partial u_k}{\partial X_i}\right) + 2(a_9 - a_2)\frac{\partial u_i}{\partial X_j}\frac{\partial u_i}{\partial X_k}\frac{\partial u_j}{\partial X_k}$$

$$+ 2(a_2 + 2a_3 - a_8 - a_9)\frac{\partial u_i}{\partial X_j}\frac{\partial u_i}{\partial X_k}\frac{\partial u_j}{\partial X_k} + 4(a_8 + 2a_7 + \tfrac{1}{3}a_9)\left(\frac{\partial u_k}{\partial X_k}\right)^3$$

$$- 2(a_8 + a_9) \frac{\partial u_j}{\partial X_j} \frac{\partial u_i}{\partial X_k} \frac{\partial u_k}{\partial X_i} + \tfrac{2}{3} a_9 \frac{\partial u_i}{\partial X_j} \frac{\partial u_j}{\partial X_k} \frac{\partial u_k}{\partial X_i} + a_4 E_i E_i + 2a_5 \frac{\partial u_i}{\partial X_j} E_i E_j$$

$$+ 2a_6 \frac{\partial u_i}{\partial X_i} E_k E_k + (2a_{10} + 4a_{11})\left(\frac{\partial u_i}{\partial X_i}\right)^2 E_k E_k - a_{10} \frac{\partial u_i}{\partial X_j} \frac{\partial u_j}{\partial X_i} E_k E_k$$

$$+ (2a_5 + a_{12}) \frac{\partial u_i}{\partial X_k} \frac{\partial u_i}{\partial X_k} E_i E_j + (2a_6 - a_{10}) \frac{\partial u_i}{\partial X_j} \frac{\partial u_i}{\partial X_j} E_k E_k + 2a_{12} \frac{\partial u_i}{\partial X_k} \frac{\partial u_k}{\partial X_j} E_i E_j$$

$$+ a_{12} \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} E_i E_j + 2a_{13} \frac{\partial u_k}{\partial X_k} \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) E_i E_j . \qquad (6.2)$$

7. Incompressible dielectric. If, in addition to being homogeneous and isotropic, the elastic dielectric considered is incompressible, the third invariant of strain I_3 is unity in all deformations, and is accordingly not a variable. The stored-energy W is then a function only of five invariants I_1, I_2, I_4, I_5, and I_6, defined in (3.7).

In terms of principal extensions e_i, the constraint of no volume change according to (4.2) demands

$$(1 + e_1)(1 + e_2)(1 + e_3) = 1. \qquad (7.1)$$

Here it is convenient to introduce a new scalar invariant J_2' :

$$J_2' = J_2 - J_1 . \qquad (7.2)$$

For the incompressible case therefore, we have instead of (5.2) the following polynomial expansion for W:

$$W = \sum_{\alpha,\beta,\delta,\lambda,\mu} B_{\alpha\beta\delta\lambda\mu} J_1^\alpha J_2'^\beta J_4^\delta J_5^\lambda J_6^\mu . \qquad (7.3)$$

With (7.1), it follows from (3.7), (4.2), (5.1), and (7.2) that the invariants J_1 and J_2' are of second and third orders, respectively, in the principal extensions. Hence, within the formulation of the first approximation for small finite deformations, defined in Sec. 5, the appropriate stored-energy-function corresponding to (5.4) is given by

$$W = b_1 J_1 + b_2 J_4 + b_3 J_5 , \qquad (7.4)$$

where b's are physical constants of the material.

For the second approximation described in Sec. 6, the form of W that corresponds to (6.1) is

$$W = b_1 J_1 + b_2 J_4 + b_3 J_5 + b_4 J_2' + b_5 J_1 J_4 + b_6 J_6 . \qquad (7.5)$$

It may be observed that in the absence of electrostatic effects, (7.4) and (7.5) reduce to the forms which in finite elasticity theory describe the well-known Neo–Hookean [8] and Mooney–Rivlin [7] materials, respectively.

In conservative systems, a pressure p arises as a reaction to the constraint of no volume change. Bearing this in mind, substitution of (7.4) in (3.9) and (3.10) leads to the following constitutive relations of the first approximation:

$$\sigma_{ij} = -p\, \delta_{ij} + C_1 g_{ij} + C_2 E_i E_j + C_3(g_{ik} E_k E_j + g_{jk} E_k E_i), \qquad (7.6)$$

$$D_i = C_2 E_i + C_3 g_{ij} E_j , \qquad (7.7)$$

where p is an arbitrary pressure and C's are the physical constants of the dielectric medium.

8. Simultaneous extension and shear of a cylindrical annulus. We consider a long tube of homogeneous, isotropic, incompressible, elastic dielectric with initial internal radius a and external radius b. The tube is subjected to a small finite deformation in the presence of a weak electric field. The particle originally located at the point (R, Θ, Z) in cylindrical polar coordinates moves to the position (r, θ, z) given by

$$r = \lambda R, \qquad \theta = \Theta + \phi(R), \qquad z = Z/\lambda^2 + w(R). \tag{8.1}$$

The above deformation is a simple extension of extension ratio $1/\lambda^2$, a simple shear of the tube in which each point moves through a distance w parallel to the axis and through an angle ϕ about the axis, both ϕ and w dependent only on the radial position of the point. It is verified readily that the mapping (8.1) is volume preserving.

We suppose that a radial electric field is imposed by placing the tube between the plates of a coaxial cylindrical condenser. In view of (2.1) the radial field has to be of the type

$$E_r = E_r(r), \qquad E_\theta = E_z = 0 \text{ inside the dielectric,} \tag{8.2}$$

and

$$E_r^{(0)} = E_r^{(0)}(r), \qquad E_\theta^{(0)} = E_z^{(0)} = 0 \text{ in the surrounding medium.} \tag{8.3}$$

Here, as in all subsequent analysis, we shall refer to physical components in cylindrical coordinate system.

The problem proposed here is to determine $E_r(r)$, $E_r^{(0)}(r)$, $\phi(R)$, $w(R)$, and the necessary surface tractions such that the deformation (8.1) can be supported without mechanical body force or distributed charge in the presence of the electric field given by (8.2) and (8.3).

In the absence of electrostatic effects, the deformation (8.1) has been studied by Rivlin [12], and Singh and Pipkin [13]. More recently, in the theory of elastic dielectrics, Singh and Pipkin [1] have shown that if the functional form of the stored-energy is regarded as arbitrary, then the deformation (8.1) can be supported without mechanical body force or distributed charge if and only if

$$\phi(R) = B \log R, \qquad w(R) = 0, \tag{8.4}$$

and the electric field is axial and uniform. However, we intend to show that if attention is confined to small finite deformations permitting the application of constitutive relations (7.6) and (7.7), a closed form solution can be obtained for $\phi(R)$, $w(R)$, $E_r(r)$, and $E_r^{(0)}(r)$. The surface tractions that can support (8.1) in the presence of field (8.2) and (8.3) can then be calculated.

With (8.1), the strain from (3.8) is

$$g_{rr} = \lambda^2, \qquad g_{\theta\theta} = \lambda^2 + r^2\phi'^2, \qquad g_{zz} = \frac{1}{\lambda^4} + w'^2,$$

$$g_{r\theta} = \lambda r\phi', \qquad g_{\theta z} = r\phi'w', \qquad g_{rz} = \lambda w', \tag{8.5}$$

where primes indicate differentiation with respect to R. To obtain the components of flux inside the dielectric, we substitute from (8.2) and (8.5) into (7.7):

$$D_r = (C_2 + C_3\lambda^2)E_r, \qquad D_\theta = C_3\lambda r\phi'E_r, \qquad D_z = C_3\lambda w'E_r. \tag{8.6}$$

Outside the dielectric, the flux components from (3.1) are simply $1/\epsilon$ times those of

the electric field (8.3). The Maxwell condition (2.2) that the flux be solenoidal both inside and outside the dielectric now provides

$$E_r = K/r, \qquad E_r^{(0)} = L/r, \tag{8.7}$$

where K and L are any constants. By virtue of the continuity condition (2.3a), $L = (C_2 + C_3\lambda^2)(K/\epsilon)$. In place of (8.2) and (8.3) therefore, we now have

$$E_r = K/r, \qquad E_\theta = E_s = 0; \qquad E_r^{(0)} = (C_2 + C_3\lambda^2)K/\epsilon r, \qquad E_\theta^{(0)} = E_s^{(0)} = 0. \tag{8.8}$$

It is immediately clear that the field (8.8) meets the continuity condition (2.3b).

The corresponding stresses follow from using (8.5) and (8.8a) into (7.6), and appear as

$$\sigma_{rr} = -p + C_1\lambda^2 + (C_2 + 2C_3\lambda^2)\frac{K^2}{r^2}, \qquad \sigma_{\theta\theta} = -p + C_1(\lambda^2 + r^2\phi'^2),$$

$$\sigma_{ss} = -p + C_1\left(\frac{1}{\lambda^4} + w'^2\right), \qquad \sigma_{r\theta} = \left(C_1 + C_3\frac{K^2}{r^2}\right)\lambda r\phi', \tag{8.9}$$

$$\sigma_{rs} = \left(C_1 + C_3\frac{K^2}{r^2}\right)\lambda w', \qquad \sigma_{\theta s} = C_1 r\phi' w'.$$

With the stresses (8.9) depending only upon r, the equations of equilibrium (2.7) become

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \qquad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} = 0, \qquad \frac{\partial\sigma_{rs}}{\partial r} + \frac{1}{r}\sigma_{rs} = 0. \tag{8.10}$$

The last two of (8.10), in conjunction with (8.9), can be integrated to yield

$$\phi(R) = A\log\frac{R^2}{F(R)} + A_1, \qquad w(R) = B\log F(R) + B_1, \tag{8.11}$$

where A, B, A_1, and B_1, are constants of integration, and to facilitate writing we have set

$$F(R) = C_1\lambda^2 R^2 + C_3 K^2. \tag{8.12}$$

Suppose we have $w = \phi = 0$ at the curved surface $r = \lambda b$, and $w = w_0$, $\phi = \phi_0$ at the curved surface $r = \lambda a$. These conditions allow us to determine the constants A, B, A_1, B_1, thus leading to

$$\phi(R) = \Psi_0\log\frac{R^2 F(b)}{b^2 F(R)}, \qquad w(R) = \Omega_0\log\frac{F(R)}{F(b)}. \tag{8.13}$$

The constants Ψ_0 and Ω_0 stand for

$$\Psi_0 = \frac{\phi_0}{\log\left(a^2 F(b)/b^2 F(a)\right)}, \qquad \Omega_0 = \frac{w_0}{\log\left(F(a)/F(b)\right)}. \tag{8.14}$$

The functions $\phi(R)$ and $w(R)$ having been so determined, we can now introduce (8.13) and (8.9) into the first of the equilibrium equations (8.10) to find the pressure p:

$$p(r) = (C_2 + 2C_3\lambda^2)\frac{K^2}{2r^2} - 2C_1\lambda^2\Psi_0^2\left[\log\frac{r^2}{C_1 r^2 + C_3 K^2} + \frac{C_3 K^2}{C_1 r^2 + C_3 K^2}\right] + \alpha, \tag{8.15}$$

where α is a constant of integration. With pressure p given by (8.15), and $\phi(R)$ and $w(R)$ given by (8.13), the stresses inside the dielectric medium are furnished by (8.9).

In the medium surrounding the dielectric, the Maxwell stresses follow by substituting the field (8.8b) into (3.2):

$$\sigma_{rr}^{(0)} = -\sigma_{\theta\theta}^{(0)} = -\sigma_{zz}^{(0)} = (C_2 + 2C_3\lambda^2)^2 K^2/2\epsilon r^2,$$

$$\sigma_{r\theta}^{(0)} = \sigma_{\theta z}^{(0)} = \sigma_{rz}^{(0)} = 0. \tag{8.16}$$

In order to support the deformation (8.1) in the field prescribed as in (8.2) and (8.3), the tractions which must be applied to the curved surfaces $r = \lambda a$ and $r = \lambda b$ are obtained from (2.8) when use is made of (8.9), (8.13), (8.15), and (8.16):

$$T_r(r = \lambda a) = (C_2 + 2C_3\lambda^2) \frac{K^2}{2\lambda^2 a^2} + 2C_1\lambda^2\Psi_0^2\left[\log \frac{\lambda^2 a^2}{C_1\lambda^2 a^2 + C_3 K^2} + \frac{C_3 K^2}{C_1\lambda^2 a^2 + C_3 K^2} \right]$$

$$- (C_2 + 2C_3\lambda^2)^2 \frac{K^2}{2\epsilon\lambda^2 a^2} + \mathfrak{G}, \tag{8.17}$$

$$T_\theta(r = \lambda a) = \frac{2C_3 K^2 \Psi_0}{a^2}, \qquad T_z(r = \lambda a) = \frac{2C_1\lambda^2 \Omega_0}{a}, \tag{8.18}$$

The constants Ψ_0 and Ω_0 above are as in (8.14). By setting the radial component T_r in (8.17) of the applied traction to zero, the arbitrary constant $\mathfrak{G}$ can be easily evaluated. Expressions for surface tractions at the curved surface $r = \lambda b$ may be derived in a similar manner replacing a by b in (8.17) and (8.18). Since we do not have accurate exyressions for the fields near the edges bounded by the plane ends of the tube, we will not calculate the end tractions. However, it is readily noted from (8.9) and (8.16) that in case the tube is of finite length, normal and shearing tractions must be applied to its ends to maintain the given state.

We observe here that if either of the curved surfaces, say $r = \lambda a$, is free from applied tangential tractions, then since the Maxwell stresses $\sigma_{r\theta}^{(0)}$ and $\sigma_{rz}^{(0)}$ happen to be zero in the field considered, the stresses $\sigma_{r\theta}$ and σ_{rz} inside the dielectric medium will also have to vanish by virtue of continuity conditions (2.8) and the last two of equilibrium equations (8.10). This, in conjunction with (8.9), would then demand $\phi'(R) = w'(R) = 0$, as to be expected.

The case of no extension is the special case of the problem considered above when λ is equal to unity. Physically, this special case may be demonstrated by holding the tube between two rigid coaxial cylinders with the outer cylinder kept fixed while moving the inner one along the axis of the tube as well as rotating it about the axis.

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