APPROXIMATIONS WITH RATIONAL POLYNOMIALS*

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Abstract. One of the cornerstones of mathematical analysis is the use of Maclaurin's series for the representation of transcendental functions. The advent of the electronic computer, however, has caused attention to be focussed on *truncated* series. A number of techniques have been developed with the object of finding the "best" approximation to a transcendental function. The present paper describes a new technique by means of which such functions can be represented with a minimum number of terms.

Introduction. It is well known that in order to represent a transcendental function (e.g. sin x or tan x) by a truncated Maclaurin's series it is necessary to take N terms in order to obtain an accuracy of $e(N)$ per cent. A standard problem in beginning calculus is to determine how many terms of a series are needed in order to keep the error below a certain amount.

In recent years some attempts have been made to increase the efficiency of the computations required to approximate a transcendental function with a finite number of terms. Lanczos [1] has proposed a method called "economization of power series" in which he determines new coefficients for the Maclaurin's series in order to obtain a greater accuracy with the same number of terms. Lanczos' method uses Chebyshev polynomials, but the integral exponents of his approximation are in the same order as they were in the Maclaurin's series.

Kovach and Comley [2] investigated the use of integral exponents which were not necessarily in the same order as those in the Maclaurin's series. They found, for example, that expressions of the form

$$ax + bx^3 + c\,|x^n|\,\mathrm{sgn}\,x \tag{1}$$

with $n = 6, 7, 8, 9$, and 10 were better approximations to tan x on $[-\pi/3, \pi/3]$ than any other three-term approximation in which the exponents are 1, 3, and 5. In all cases the coefficients a, b, and c were determined so that the full-scale error curve

$$E(x) = 3^{-1/2}(ax + bx^3 + c\,|x^n|\,\mathrm{sgn}\,x - \tan x)$$

had equal maximum and minimum values. We call such a curve "an optimized error curve."

Using the methods described in [2], one can obtain the approximation

$$\tan x \doteq 0.99845x + 0.37236x^3 + 0.18752x^7 \tag{2}$$

having maximum full-scale error of 0.00054 on $[-\pi/3, \pi/3]$. Approximation (2) yields the smallest error of all the approximations shown in (1).

Rational polynomials. The next natural question is whether the error in (2) can be reduced by using *rational* exponents. Accordingly, we define a *rational polynomial* in x as an expression of the form

$$\sum_{i=1}^{n} a_i x^{\alpha_i} \tag{3}$$

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where the α_i and a_i are rational numbers and

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n .$$

Thus rational polynomials are a generalization of polynomials and (not surprisingly) have some interesting properties.

In order to show the effect of approximating by rational polynomials, a step-by-step method was used. First, the approximation

$$\tan x \doteq 0.99483x + 0.38601x^3 + 0.17724\ |x|^{7.2} \operatorname{sgn} x \tag{4}$$

was found to have a maximum error of 0.00039 on $[-\pi/3, \pi/3]$, an improvement of the 0.00054 error of (2). Second, the approximation

$$\tan x \doteq 1.0034x + 0.43466\ |x|^{3.3} \operatorname{sgn} x + 0.11951\ |x|^{8.3} \operatorname{sgn} x \tag{5}$$

was found to have a maximum error of 0.00015. By using one more rational exponent the error was reduced to less than one-half of the previous value!

Method of computation. The coefficients of the rational polynomial approximation

$$F(x) \doteq \sum_{i=1}^{n} a_i x^{\alpha_i} \tag{6}$$

on the interval $[0, r_n]$ may be defined by the n simultaneous linear equations

$$\sum_{i=1}^{n} a_i e_j^{\alpha_i} = F(r_j), \qquad i = 1, 2, \cdots, n. \tag{7}$$

The exponents α_i and intermediate points of zero error r_j, $j = 1, 2, \cdots, n - 1$, on $(0, r_n)$ are varied empirically to obtain an optimized error curve.

Another method makes use of the principle of least squares. In the least squares sense, the "best" approximation to $F(x)$ for a given set of exponents in (6) is the one for which the sum

$$S = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_j x_i^{\alpha_j} - F(x_i) \right]^2 \tag{8}$$

is a minimum. Setting the partial derivatives $\partial S/\partial a_j$, $j = 1, 2, \cdots, n$, equal to zero, the n simultaneous linear equations

$$\sum_{j=1}^{n} \left(a_j \sum_{i=1}^{m} x_i^{\alpha_k + \alpha_j} \right) = \sum_{i=1}^{m} F(x_i) x_i^{\alpha_k} \qquad (k = 1, 2, \cdots, n) \tag{9}$$

are readily obtained. For a suitable m, the least squares approach leaves only the n exponents of (6) for empirical study.

Approximations to $\sin x$. If the Maclaurin series representation of $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \tag{10}$$

is truncated after four terms, the resulting polynomial approximation to $\sin x$, if limited to $[-1, 1]$, can be reduced to the binomial approximation

$$\sin x \doteq 0.9974826x - 0.1565104x^3 \tag{11}$$

having maximum simple[1] error less than 0.00051 by Lanczos' method of "economization of power series" [3]. However, the rational binomial approximation to sin x over $[-1, 1]$,

$$\sin x \doteq 1.0053607\ |x|^{1.0022} \operatorname{sgn} x - 0.1638897\ |x|^{2.8628} \operatorname{sgn} x, \tag{12}$$

has maximum simple error less than 0.000075, or nearly *one-seventh* the error obtained by Lanczos' "economization" process. (See Fig. 1.)

Hastings [4] has given the approximation

$$\sin \pi x/2 \doteq 1.5706268x - 0.6432292x^3 + 0.0727102x^5 \tag{13}$$

which has a maximum relative error less than 0.00011 on $[-1, 1]$. The rational trinomial approximation

$$\sin \pi x/2 \doteq 1.5708268x - 0.6478298x^3 + 0.0770030\ |x|^{4.85} \operatorname{sgn} x \tag{14}$$

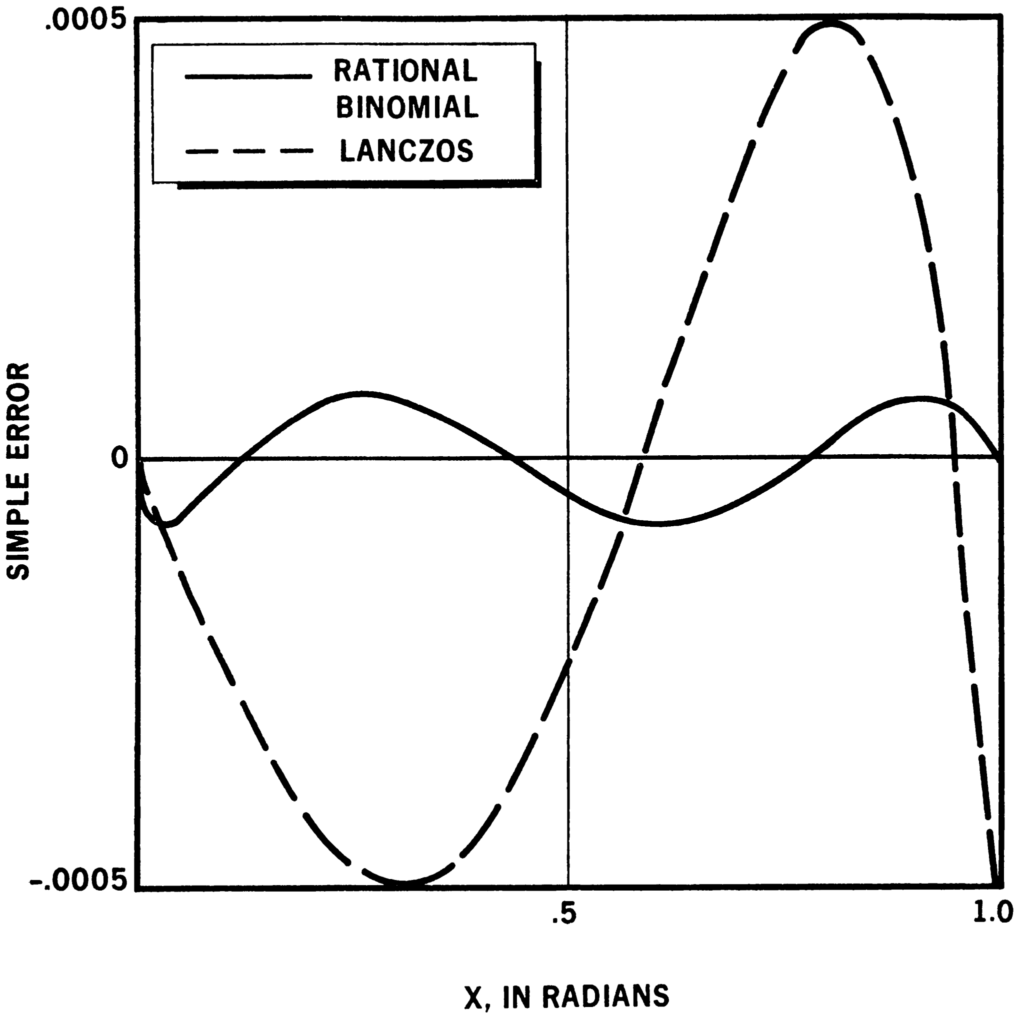

FIG. 1. Rational binomial approximation of sin x compared with two-term approximation by Lanczos' economization of power series.

[1]Simple error is defined as |approximation—true value|.

has a maximum relative error of 0.000022 or about *one-fifth* the error of the Hastings approximation. (See Fig. 2.)

Other rational polynomial approximations. In the course of this work other approximations were found and some of these are given to illustrate the properties of rational exponents. These are, for the interval $[-\pi/3, \pi/3]$,

$$\tan x \doteq 0.95602x + 0.25161x^2 + 0.35200\,|x|^{5.5684}\,\mathrm{sgn}\,x \tag{15}$$

with maximum full-scale error 0.0013,

$$\tan x \doteq 1.6055\,|x|^{1.6456}\,\mathrm{sgn}\,x \tag{16}$$

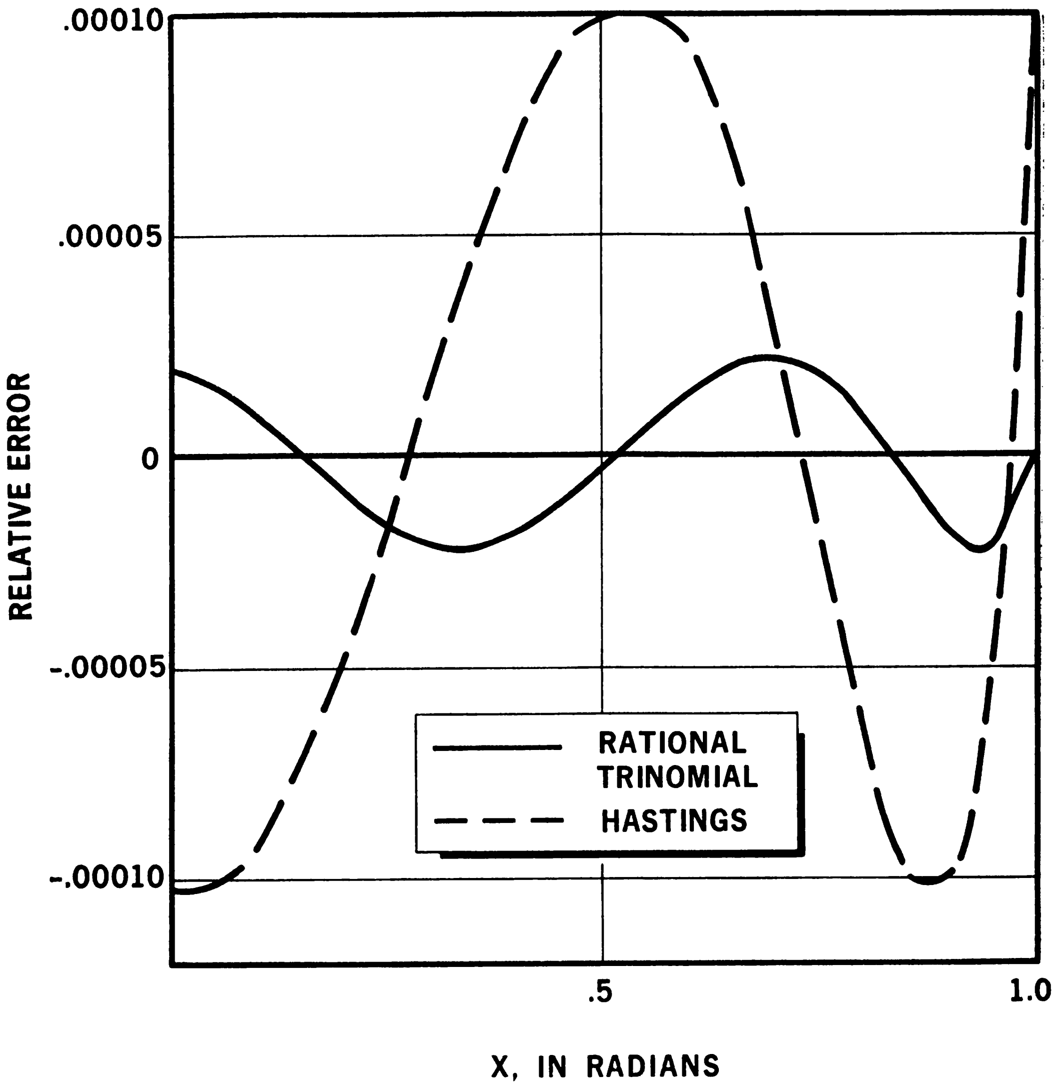

Fig. 2. Rational trinomial approximation of sin $\pi x/2$ compared with three-term approximation by Hastings.

with maximum full-scale error 0.053, and

$$\tan x \doteq 1.1141\ |x|^{1.06}\operatorname{sgn} x + 0.4468\ |x|^{4.98}\operatorname{sgn} x \tag{17}$$

with maximum full-scale error of 0.0021.

An advantage to using rational polynomials appears to be the fact that the error curves have *more* zeroes. It was found, for example, that approximating tan x with a rational monomial resulted in an error curve which had one zero (in addition to those at zero and the end-points of the approximating interval). Similarly, approximating tan x with a rational binomial resulted in an error curve having *three* zeros in addition to those mentioned above.

Conclusion. Approximations with rational polynomials can produce spectacular results. The most important of these is that the use of rational polynomials results in considerably less error than methods using the long-revered Chebyshev polynomials [5].

On the other hand, rational polynomials may not have immediate practicability because the logic of present-day computers is oriented toward integral exponents. Thus it may be cheaper from the standpoint of computer time to use polynomials rather than rational polynomials to obtain equivalent accuracy. A fair comparison between the two methods cannot be made at present.

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Bibliography

[1] C. Lanczos, *Applied analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1956

[2] L. D. Kovach and W. Comley, *A unique approach to the approximation of trigonometric functions*, Amer. Math. Monthly (68) 9, 839–846 (1961)

[3] F. B. Hildebrand, *Introduction to numerical analysis*, McGraw-Hill, New York, 1956

[4] C. Hastings, Jr., *Approximations for digital computers*, Princeton University Press, Princeton, N. J., 1955

[5] C. W. Clenshaw, *Chebyshev series for mathematical functions*, Math. Tables No. 5 N.P.L., London HMSO, 1962