TRANSITION RADIATION AND THE ČERENKOVA EFFECT*

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A. Introduction. When a charge crosses the boundary between two regions with different electrical properties a burst of electromagnetic energy is radiated. This phenomenon is called transition radiation.

In a previous paper [1] the authors considered the problem of transition radiation of a line charge. The problem considered is illustrated in Fig. 1. As shown in Fig. 1, the line charge (of charge $Q$ per unit length) moves with a constant velocity $v$ and crosses a boundary between two dielectrics at the time $t = 0$. The dielectric constant is $\epsilon_+$ for $z > 0$ and $\epsilon_-$ for $z < 0$. The magnetic permittivity is $\mu_+$ for $z > 0$ and $\mu_-$ for $z < 0$.

In [1], $v$ was assumed to be less than the speed of light in both media, i.e.

$$v < c_+,$$

and

$$v < c_-,$$

where

$$c_\pm = (\mu_\pm \epsilon_\pm)^{-1/2}.$$

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Under these conditions no Čerenkov radiation is produced. In this paper we wish to consider the interesting problem of what happens when $v$ is greater than the speed of light in either one or both of the media.

**B. General Considerations.** In general there are six cases:

<table>
<thead>
<tr>
<th>case</th>
<th>$c_+/c_-$</th>
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**Table I.**

Cases 1 and 2 have already been dealt with in [1].

In case 3 the charge goes from a medium in which it produces no Čerenkov radiation to a medium in which it does. The solution for this case demonstrates how the Čerenkov radiation establishes itself after the impact time at $t = 0$.

In case 4 a charge producing Čerenkov radiation crosses into a region in which it produces no Čerenkov radiation. The situation for $t < 0$ is shown in Fig. 2.

![Fig. 2. Čerenkov radiation for $t < 0$.](image)

The angle of the Čerenkov wedge is

\[ \theta_- = \arccos \left( \frac{c_-}{v} \right). \]  

After impact one expects the wavefronts to act like infinite plane waves incident on the boundary $z = 0$. For $\theta_-$ less than the critical angle, $\theta_\epsilon$ ($\theta_\epsilon = \arcsin \left( \frac{c_+}{c_-} \right)$), it is suspected that the incident Čerenkov wavefronts will produce transmitted and reflected wavefronts. For $\theta_- > \theta_\epsilon$, critical reflection should occur with no transmitted wave produced.

In cases 5 and 6 the charge produces Čerenkov radiation in both media. One expects the phenomena taking place in these two situations essentially to be combinations of those occurring in cases 3 and 4.
C. Analysis. In all of the following it will be assumed that the reader is familiar with the contents and notation of [1].

In [1] the potential function $F$ was expressed as

$$F = F_p + F^{(1)} + F^{(2)} + F^{(3)}.$$  \hfill (2)

The expressions for $F_p$, $F^{(1)}$, and $F^{(2)}$ given in that paper are valid for all cases. The expressions for $F^{(3)}$ (the pole contributions) must be evaluated separately for each case. Thus the remainder of this work will be concerned with evaluating $F^{(3)}$ for each of the four remaining cases.

1. Case 3. The disposition of singularities and integration path in the $w$-plane is shown in Figs. 3, 4 and 5, below, for case 2.

![Fig. 3. Poles of $A_\prime(w)$, case 3.](image)

![Fig. 4. Poles of $A_\prime(w)$, case 3, $w_{p1} < w_{b1}$.](image)

After deforming the integration path [1] we see that the pole at $\pm w_{p1}$ contributes if $|\theta| > w_{p1}$. The values of the pole contributions from $\pm w_{p2}$ and $\pm w_{p3}$ are precisely the same as in case 1.

For $w_{p1} < w_{b1}$, the contribution from $\pm w_{p1}$ is a simple residue term. This term is such that it cancels out $F_p$ in $z > 0$, $|\theta| > w_{p1}$. The resulting wavefront diagram is shown in Fig. 6.
For \(w_{p1} > w_{h1}\), Fig. 5 applies. The pole \(w_{p1}\) lies on the branch cut. The deformed integration path is shown in Fig. 7. In this case, the contribution from \(w_{p1}\) is the sum of one half the residues on the top and bottom of the branch cut. The branch cut integral \(I^{(2)}\) is taken in the sense of a principal part at \(w_{p1}\). It again turns out that the contribution from \(w_{p1}\) cancels \(F_\sigma\) for \(z > 0, |\theta| > w_{p1}\). The resulting wavefront diagram is shown in Fig. 8.

\[\text{Fig. 5. Poles of } A_\sigma(w), \text{ case 3, } w_{p1} > w_{h1}.\]

\[\text{Fig. 6. Wavefront diagram, case 3, } w_{p1} < w_{h1}.\]

The analytical expressions for \(F^{(3)}\) are given below.

For \(z < 0\)

\[F^{(3)} = \frac{- (1 + \beta_+^2 - \beta_-^2)^{1/2} - \epsilon Q \arctan \left\{ \frac{z + vt}{x(1 - \beta_-^2)^{1/2}} \right\}}{(1 + \beta_+^2 - \beta_-^2)^{1/2} + \epsilon 2\pi}, \tag{3a}\]

where \(\beta_\pm = v/c_\pm\) and \(\epsilon = \epsilon_+ / \epsilon_-\).
Fig. 7. Integration path for $\theta > \omega_{pl} > \omega_{bl}$.

Fig. 8. Wavefront diagram, case 3, $\omega_{pl} > \omega_{bl}$.

Fig. 9. Poles of $A_-(w)$, case 4, $\theta_+ < \theta_c$. 
For $z > 0$

$$F^{(3)} = \sgn(x) \frac{Q}{2} u(vt - z - |x| [\beta_+^2 - 1]^{1/2})u(|\theta| - \cos^{-1} [\epsilon/v])$$

$$- \frac{2\epsilon}{(1 + \beta_+^2 - \beta_-^2)^{1/2} + \epsilon} \frac{Q}{2\pi} \arctan \left( \frac{z(1 + \beta_+^2 - \beta_-^2)^{1/2} - vt}{x(1 - \beta_-^2)^{1/2}} \right)$$

where $u(t)$ denotes the unit step.

Figures 6 and 8 illustrate how the Čerenkov radiation establishes itself in the region $z > 0$ after impact.

2. Case 4. The location of poles for $\theta_+ > \theta_-$ is shown in Figs. 9 and 10. (Note: $\theta_+ = \omega_{b_1}$, $\theta_- = \omega_{p_2}$.) When $\theta_+ < \theta_-$ the pole configuration is as shown in Figs. 11 and 12. We see that for $\theta_- > \theta_+$ the pole $\omega_{p_3}$ will never be intercepted. For $\theta_- < \theta_+$, $\omega_{p_3}$ yields a transmitted wavefront in $z > 0$. The pole $\omega_{p_2}$ yields a reflected wavefront. The wavefront diagrams are shown, below, in Figs. 13 and 14.

For $\theta_- < \theta_+$,

$$F^{(3)} = \frac{\epsilon - (1 + \beta_+^2 - \beta_-^2)^{1/2}}{\epsilon + (1 + \beta_+^2 - \beta_-^2)^{1/2}} \frac{Q}{2} \sgn(x)u(|\theta| - \theta_-)u(vt - z - |x| [\beta_-^2 - 1]^{1/2}), \quad z < 0$$

(4a)
Fig. 12. Poles of \( A_+(w) \), case 4, \( \theta_- > \theta_c \).

Fig. 13. Wavefronts for case 4, \( \theta_- < \theta_c \).

Fig. 14. Wavefronts for case 4, \( \theta_- > \theta_c \).
and

\[ F^{(3)} = \frac{2\varepsilon}{\varepsilon + (1 + \beta^2_+ - \beta^2_-)^{1/2}} \cdot \frac{Q}{2} \text{sgn} (x) u\left( |\theta| - \arcsin \left( \frac{c_-}{c_+} - \frac{1}{\beta^2_+} \right)^{1/2} \right) \]

\[ \cdot u(\nu t - z[1 + \beta^2_+ - \beta^2_-]^{1/2} + |x| [\beta^2_+ - 1]^{1/2}) + \frac{Q}{2\pi} \arctan \left( \frac{z - v t}{x(1 - \beta^2_+)^{1/2}} \right), \quad z > 0. \quad (4b) \]

The factors

\[ \frac{\varepsilon - (1 + \beta^2_+ - \beta^2_-)^{1/2}}{\varepsilon + (1 + \beta^2_+ - \beta^2_-)^{1/2}} \quad \text{and} \quad \frac{2\varepsilon}{\varepsilon + (1 + \beta^2_+ - \beta^2_-)^{1/2}} \]

are just the Fresnel reflection and transmission coefficients for an infinite plane wave incident at an angle \( \theta_- \).

For \( \theta_- > \theta_e \)

\[ F^{(3)} = \frac{1 + \beta^2_+ - \beta^2_- + \varepsilon \cdot Q}{1 + \beta^2_+ - \beta^2_- - \varepsilon \cdot 2} \text{sgn} (x) u(|\theta| - \text{arc sec} \beta_-) u(\nu t - z - |x| [\beta^2_- - 1]^{1/2}), \]

\[ z < 0, \quad (5a) \]

and

\[ F^{(3)} = \frac{Q}{2\pi} \arctan \left( \frac{z - v t}{x(1 - \beta^2_+)^{1/2}} \right), \quad z > 0. \quad (5b) \]

3. Case 5. For cases 5 and 6 only the results will be given.

For Case 5, \( F^{(3)} \) is given by

\[ F^{(3)} = \frac{\varepsilon - (1 + \beta^2_+ - \beta^2_-)^{1/2}}{\varepsilon + (1 + \beta^2_+ - \beta^2_-)^{1/2}} \cdot \frac{Q}{2} \text{sgn} (x) u(|\theta| - \text{arc sec} \beta_-) u(\nu t + z - |x| [\beta^2_- - 1]^{1/2}), \]

\[ z < 0 \quad (6a) \]

and

\[ F^{(3)} = -\frac{Q}{2} \text{sgn} (x) \left\{ u(\nu t - z - x[\beta^2_+ - 1]^{1/2}) u(|\theta| - \text{arc sec} \beta_+) \right. \]

\[ \left. - \frac{2\varepsilon}{\varepsilon + (1 + \beta^2_+ - \beta^2_-)^{1/2}} u(\nu t - z[1 + \beta^2_+ - \beta^2_-]^{1/2} - |x| [\beta^2_- - 1]^{1/2}) \right. \]

\[ \left. \cdot u\left( |\theta| - \arcsin \left( \frac{c^2_+}{c^2_-} - \frac{1}{\beta^2_+} \right)^{1/2} \right) \right\}, \quad z > 0. \quad (6b) \]

The wavefront diagrams are shown in Figs. 15 and 16.

4. Case 6. For \( z < 0 \) and \( \theta_- < \theta_e \),

\[ F^{(3)} = \frac{\varepsilon - (1 + \beta^2_+ - \beta^2_-)^{1/2}}{\varepsilon + (1 + \beta^2_+ - \beta^2_-)^{1/2}} \cdot \frac{Q}{2} \text{sgn} (x) u(|\theta| - \theta_-) u(\nu t + z - |x| [\beta^2_- - 1]^{1/2}) \quad (7a) \]
For $z > 0$ and $\theta_- < \theta_+$,

$$F^{(3)} = -\frac{Q}{2} \text{sgn}(x) \left\{ u(vt - z - x[\beta_+^2 - 1]^{1/2}) u(|\theta| - \text{arc sec} \beta_+) \right. $$

$$- \frac{2\epsilon}{\epsilon + (1 + \beta_+^2 - \beta_-^2)^{1/2}} u(vt - z[1 + \beta_+^2 - \beta_-^2]^{1/2} - |x| [\beta_+^2 - 1]^{1/2})$$

$$\cdot u(|\theta| - \text{arc sin} \left[ \frac{c^2}{c_+^2} - \frac{1}{\beta_-^2} \right]^{1/2}) \right\}. \quad (7b)$$

**Fig. 15.** Case 5, $w_{pl} > w_{bl}$.

**Fig. 16.** Case 5, $w_{pl} < w_{bl}$.
For $z < 0$ and $\theta_+ > \theta_-$,

$$P^{(3)} = \frac{1 + \beta_+^2 - \beta_-^2 + \epsilon}{1 + \beta_+^2 - \beta_-^2 - \epsilon} \frac{Q}{2} \text{sgn}(\nu)(|\theta| - \theta_+)u(\nu t + z - |x| [\beta_+^2 - 1]^{1/2}).$$  \hspace{2cm} (8a)

For $z > 0$ and $\theta_+ > \theta_-$,

$$P^{(3)} = -\frac{Q}{2} \text{sgn}(\nu)(\nu t - z - |x| [\beta_+^2 - 1]^{1/2})u(|\theta| - \text{arc sec } \beta_+).$$  \hspace{2cm} (8b)

The wavefront diagrams for case 6 appear in Figs. 17 and 18.

**Fig. 17.** Case 6, $\theta_+ > \theta_-$.

**Fig. 18.** Case 6, $\theta_- < \theta_+$.

**Reference**