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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ('), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\quad}$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi x / 2 b)}{\cos (\pi a / 2 b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2 b}}{\cos \frac{\pi a}{2 b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t (a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{(a + (b + cx)^n) \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

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The following examples show the desired arrangements: (*for books*)—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

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—BOOK REVIEW SECTION—

The eightfold way. By Murray Gell-Mann and Yuval Ne'eman. W. A. Benjamin, Inc., New York, Amsterdam, 1964. xi + 317 pp. \$9.00.

One of the most exciting steps in our exploration of the properties of the strong nuclear interactions which are responsible for the existence of the "elementary particles" has been the recognition that these interactions are invariant under symmetry operations which relate physically distinct particle states, and which therefore represent a symmetry lying beyond the properties of space-time. The largest group of transformations for which this has been established experimentally are those of the unitary symmetry $SU(3)$, although there are some indications that such an invariance may hold for some still larger group.

This $SU(3)$ invariance property does not hold exactly. It is violated by the "moderately-strong interactions", which are characterised typically by energy contributions of order 200 MeV (which may be compared with the proton rest energy, about 940 MeV). However, even these latter interactions are invariant with respect to the $SU(2)$ subgroup of isospin transformations, which invariance gives rise to the property of "charge-independence" well known in nuclear physics. This $SU(2)$ invariance is violated mainly by the electromagnetic interactions (necessarily charge-dependent, since they serve to define charge) and gives rise to the occurrence of all strongly-interacting particle states (the hadronic states) as isospin multiplets, groups of states ($(2I + 1)$ of them for isospin I) with the same spin-parity and the same hypercharge Y , with essentially the same mass but with differing charge values (running in integral steps from $(-I + \frac{1}{2}Y)e$ to $(I + \frac{1}{2}Y)e$). The $SU(3)$ symmetry is reflected by the grouping of the hadronic states into unitary multiplets, consisting of characteristic patterns of isospin multiplets (I, Y) with the same spin-parity and with comparable mass values (mass differences typically 200 MeV within the unitary multiplet). The unitary multiplets known physically are singlet, octet and decimet, corresponding to the 1-, 8-, and 10-dimensional representations of the $SU(3)$ group. The particles we know best are those from the baryon octet (spin-parity $(\frac{1}{2}^+)$, including the (proton, neutron) isospin doublet) and the pseudoscalar meson octet (spin-parity (0^-) , including the pion triplet), and it is from the octet patterns associated with these particles that the term "Eightfold Way" has come to be used for this $SU(3)$ invariance.

This book is a collection of basic papers about the development of the $SU(3)$ symmetry scheme for the elementary particles. These papers are grouped in chapters, each prefaced by an editorial introduction which sets the scene and refers the reader to parallel papers of interest and to later developments. The collection includes Gell-Mann's unpublished (but widely circulated) paper of 1961, "The Eightfold Way: A Theory of Strong Interaction Symmetry", together with Ne'eman's pioneering paper which suggested that the $SU(3)$ group may be relevant to mesons and baryons. There are chapters on the mass formula appropriate for the substates of a unitary multiplet, with broken $SU(3)$ symmetry, and on the assignment of mesonic and baryonic states to $SU(3)$ representations, including the effect of mixing between different $SU(3)$ multiplets due to the symmetry-breaking interactions. The $SU(3)$ Clebsch-Gordan tables of de Swart are reprinted here, together with some applications.

The properties of the electromagnetic interaction and of the weak currents associated with the weak decay processes for hadrons are discussed in some detail, including particularly the Coleman-Glashow formula for the electromagnetic mass-differences and the Cabibbo $SU(3)$ scheme for the weak currents. In these chapters, there is an emphasis on the physical significance of currents and of the algebra generated by them, which was a forewarning of the rapid development we have seen in the application of current-algebra methods to elementary particle physics over the last few years. For the non-leptonic decay processes, the need for some "Octet Enhancement" mechanism is discussed, including d'Espagnat's attempt to account for this by assigning $SU(3)$ properties to the hypothetical W mesons, the carrier fields for the weak interactions. There is also a chapter of papers about the bootstrap approaches to the questions of the stability of $SU(3)$ symmetry and the origin of the symmetry-breaking.

Although there has been a great increase over the past few years both in the range of applications of $SU(3)$ symmetry and in the experimental data bearing on this symmetry, this collection of papers still provides a convenient introduction to the central topics in this field of physics.

R. H. DALITZ (*Oxford, England*)

Praktische Funktionenlehre. By Friedrich Tölke. Band 2. (*Theta-Funktionen und spezielle Weierstrassche Funktionen*). Springer-Verlag, Berlin, Heidelberg, New York, 1966. vii + 248 pp. \$21.15. Band 3. (*Jacobische elliptische Funktionen, Legendresche Normalintegrale, und spezielle Weierstrassche Zeta- und Sigma-Funktionen*). Springer-Verlag, 1967. viii + 180 pp. \$17.37.

These are two volumes of a work presenting special functions in a manner suitable for the practical applied mathematician. Elaborate and explicit lists of formulas and numerous diagrams drawn to scale fill much of these volumes. Proofs, mostly by calculation rather than theoretical principles, are given or indicated. The mathematical presentation is governed by the aim of the work and is generally clear and simple. Some regrettable lapses do occur, generally when the author attempts to be more precise or more sophisticated than circumstances warrant. For instance, $\vartheta_2(\xi, \kappa)$ is said to vanish for real ξ when $\kappa = 0$, except for integer values of ξ where it becomes singular "according to

$$\vartheta_2(n, 0) = (-1)^n \lim_{\Delta\xi \rightarrow 0} \frac{1}{\Delta\xi}$$

(vol. 2, p. 2), and the series

$$\vartheta_2(\xi, \kappa) = 2 \sum_{n=0}^{\infty} (-1)^n \exp[-(n + \frac{1}{2})^2 \pi \kappa] \cos(2n + 1)\pi\xi$$

is stated (and "proved") to be uniformly convergent for $-\infty < \xi < \infty$ and $\kappa > 0$ (instead of $\kappa \geq \delta > 0$) (vol. 2, p. 4). Such faults, however, do not influence the practical usefulness of the work.

Altogether four volumes of the series (vols. 2-5) will be devoted to elliptic functions and related functions, with a fifth (vol. 6) of numerical tables.

In vol. 2, theta functions are introduced as solutions of the partial differential equation

$$\frac{\partial^2 \vartheta}{\partial \xi^2} - 4\pi\kappa \frac{\partial \vartheta}{\partial \kappa} = 0.$$

For instance, ϑ_2 is that solution behaving for $\kappa = 0$ as stated above, and from this initial condition, the series expansion given above is "proved". The parameter κ is connected with that used in mathematical theory by $\kappa = -ir$. In addition to the classical functions ϑ_1 to ϑ_4 , two further functions

$$\vartheta_5(\xi, \kappa) = \vartheta_1\left(\xi + \frac{1}{4}, \frac{\kappa}{2}\right) \pm \vartheta_1\left(\xi - \frac{1}{4}, \frac{\kappa}{2}\right)$$

are introduced. These enable the author to represent the product of two theta functions as a multiple of a single theta function. The modulus k and the complete elliptic integrals E and K are defined in terms of theta functions, and the representation

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-1/2} d\phi$$

is derived from this definition. The e_α of the Weierstrass theory are also represented in terms of theta functions.

Altogether six "Weierstrass functions" are defined by

$$\gamma_\alpha(\xi, \kappa) = c_\alpha - \frac{1}{\pi^2 \vartheta_3^3(0, \kappa)} \frac{\partial^2}{\partial \xi^2} \log \vartheta_\alpha(\xi, \kappa),$$

where

$$\begin{aligned} c_\alpha &= e_1 - \frac{E}{K} \quad \text{for } \alpha = 1, 2, 3, 4, \\ &= 2e_1 - e_2 - \frac{2E}{K} \quad \text{for } \alpha = 5, 6. \end{aligned}$$

In vol. 3, Jacobian functions are introduced through quotients of theta functions. In addition to the 12 functions pqz , where p and q stand for any two of s, c, d, n , the 12 further functions $pq = \partial/\partial z \log pq$ are also introduced and numerous relations are given involving these 24 functions, and also the Weierstrass functions. Legendre's canonical elliptic integrals appear as inverse functions of Jacobi functions, and vol. 3 concludes with two brief chapters, respectively on Weierstrass's zeta and sigma functions (6 of each).

Even this brief indication of the contents of these two volumes shows that the presentation leans heavily on theta functions—perhaps because they lend themselves to rapid computation. By comparison, the transformation theory (which can also be exploited to accelerate computation) is underplayed. There is an almost overwhelming wealth of material in these volumes: in 428 pages they contain 1082 numbered equations and 224 figures. There are, of course, formulas without numbering, and some of the numbers designate groups of as many as 12 formulas; and some of the figures consist of half a dozen diagrams or several families of graphs. The multitude of formulas includes not only expansions, functional equations, and the like but, most usefully, also approximations with stated accuracy; however, “rational approximations” of the kind used in modern large-scale computing are not included. There is a detailed bibliography in each of the two volumes, but no index.

Elliptic functions and integrals present a notoriously intractable problem of notation and organization. Many have tried, without fully succeeding, to devise a system which exhibits the inherent symmetry, is practical, and not too complicated. The author has gone further in this direction than others: the success of his system remains to be seen. The large number of functions he uses results in a proliferation of formulas which presumably pays dividends in extensive and constant use but may make orientation difficult to the casual user who learned his elliptic functions from text-books. Assistance to the latter class of users is provided by frequent cross references.

A. ERDÉLYI (*Edinburgh Scotland*)

Integral equations and their applications. By W. Pogorzelski. Volume I. Pergamon Press, New York, and Polish Scientific Publishers, Warsaw, 1966. xiii + 714 pp. \$18.50.

Witold Pogorzelski died in 1963 at the age of 67, leaving this book as the culmination of his long-continued research in integral equations and their applications. The three parts of the work correspond to the three separate volumes of the Polish edition. Part I contains a thorough account of the classical theory of nonsingular integral equations, but offers little in the way of novelty; the real interest of the book is in the later Parts.

Part II is concerned with systems of integral equations, nonlinear integral equations, and applications to differential equations. The treatment of nonlinear equations includes the classical methods of successive approximation, and the use of the Banach and Schauder fixed-point theorems to obtain existence and (in the Schauder case, with the help of auxiliary arguments) uniqueness theorems. An appendix (by R. Sikorski) contains a proof of Schauder's theorem. The short chapter on ordinary differential equations outlines the usual material on initial-value and boundary-value problems, and contains an interesting section on periodic solutions.

The long chapter on elliptic partial differential equations goes well beyond the standard treatment of the Dirichlet and Neumann (wrongly translated as “von Neumann”) problems; in particular, the author discusses nonlinear equations and nonlinear boundary conditions, making much use of Schauder's theorem, and obtains the fundamental solutions for general linear elliptic equations. The next chapter deals with parabolic equations, and includes the author's own discussion of fundamental solutions for equations of this class. The chapter on hyperbolic equations contains, in addition to an account of the more familiar methods involving the Riemann function, Vekua's method of transforming an analytic elliptic equation into a formally hyperbolic equation in complex variables, and using Riemann-function techniques on the latter to get information about the former.

Part III is devoted to singular integral equations involving Cauchy principal-value integrals. There is a thorough treatment of the standard Muskhelishvili-Vekua theory; the applications discussed include problems with boundary conditions involving tangential derivatives and questions concerning multiply connected domains. An important chapter on nonlinear singular equations includes many of the author's own results; fixed-point theorems again play an important role, and there are applications

to generalized Hilbert and Riemann problems with nonlinear boundary conditions. The last chapter is concerned with discontinuous boundary-value problems and problems involving nonclosed arcs, which are particularly important in elasticity theory; the author goes beyond the usual theory, showing how one can deal with cases where a number of arcs meet at a point, and also discussing nonlinear problems.

At the time of his death Pogorzelski had just completed a fourth Part, which is being prepared for publication; this will include polyharmonic equations, multidimensional singular integral equations, and a number of other topics, including some applications to problems from physics and engineering.

The translation of Parts II and III is unidiomatic but adequate, but that of Part I leaves a good deal to be desired; some technical terms are mistranslated, and many phrases are so un-English in style as to be difficult to understand.

The book provides a comprehensive and valuable conspectus of the many ways in which integral equations find application in physical and engineering problems; in particular, the power of Schauder's theorem in the treatment of nonlinear problems is well exhibited.

F. SMITHIES (*Cambridge, England*)

Physics of shock waves and high-temperature hydrodynamic phenomena. By Ya. B. Zel'dovich and Yu. P. Raizer. Edited by W. D. Hayes and R. F. Probst. Volume I. Academic Press, New York, London, 1966. xxiv + 464 pp. \$18.00.

This volume and another volume to follow are intended as a comprehensive treatment of shock theory and shock related effects. A look at the chapter headings (I. Elements of gas-dynamics and the classical theory of shock waves, II. Thermal radiation and radiant heat exchange in a medium, III. Thermodynamic properties of gases at high temperatures, IV. Shock tubes, V. Absorption and emission of radiation in gases at high temperatures, VI. Rates of relaxation processes in gases) reveals the wide front on which shock wave physics is examined. In addition this book is so up-to-date that it contains accounts of plasma and laser effects, and references as recent as 1966.

For both the clarity of exposition and the technical detail this is a remarkably successful book. The main emphasis is on physical understanding and this is achieved in an easy and relaxed style. The editors are to be complimented on bringing forth such an important and readable translation.

L. SIROVICH (*Providence, R.I.*)

The theory of splines and their applications. By J. H. Ahlberg, E. N. Nilson and J. L. Walsh. Academic Press, New York, London, 1967. xi + 284 pp. \$13.50.

This is the first book on splines to appear since Schoenberg first introduced the concept of a mathematical spline in 1946. The considerable activity in the theory of splines has uncovered such a multitude of mathematical properties that a compendium of these results is overdue.

There has long been a need for material on spline applications, both because of the considerable activity in the field and the lack of attention it has received in published spline papers. Spline theory had its origins in that old but still practical device, the draftsman's spline, and the association with practical applications is still very strong. In fact, it has been the rapidly expanding, computer-oriented application of cubic splines to practical curve-fitting problems at industrial organizations, notably United Aircraft (as in the case of the authors) and General Motors, which has motivated and supported much of the recent theoretical research. Considering this association, it is remarkable that most of the published papers on splines have not only minimized practical results but have frequently left the erroneous feeling that splines were numerically difficult. (For example, the $x_{\frac{1}{2}}$ notation has been quite popular even though it is computationally useless.)

Chapter One is recommended to everyone, as it contains both a brief description of a spline and an historical summary of theoretical properties which is particularly lucid and informative. Applications-minded computer programmers and engineers are not likely to stray past Chapter Two (The Cubic Spline), but that one-fourth of the book will amply reward their interest. In presenting the cubic spline equations, the authors have nicely handled the usually awkward collection of end conditions and, by deriving both sets of equations, have avoided making a premature choice between slopes and moments as unknowns.

Section 2.6 is an excellent two-page discussion of curve fitting which could (and should) have been expanded to at least ten pages, considering the authors' extensive practical experience and the fact that this has been the most fruitful area for spline applications. It also would have been the logical place for the authors to collect, for easy reference, their recommendations on end conditions, that all-important choice in the art of spline fitting. Section 2.6 also deserves special attention since it elucidates the seldom-seen-in-print but highly successful method of discrete parametric representation; the authors discuss fitting splines to the parametrically displayed data, but, of course, any suitable function can be used. A serious omission for those interested in applications is the lack of any discussion of the practical implementation of de Boor's doubly cubic spline surface, which has seen important industrial application.

The remaining three-fourths of the book is devoted to a well-organized presentation of much of the published material on the theory of splines as well as a considerable amount of previously unpublished work. There is a systematic presentation of existence, uniqueness, convergence, and best approximation properties for the cubic, polynomial, and generalized one-dimensional splines and the cubic and generalized two-dimensional splines. The extent of the material and the eminence of the authors make this book a milestone in the development of the theory of splines.

DANIEL G. SCHWEIKERT (*Murry Hill, N.J.*)

Constructive real analysis. By Allen A. Goldstein. Harper & Row, Publishers, New York, Evanston, London, 1967. xii + 178 pp. \$9.25.

This text is a thoroughly modern, lucidly written introduction to a variety of topics of functional analysis and their use in dealing with various algorithms for finding zeros and extremals of functions with and without constraints. By applying techniques of "soft" analysis to problems of "hard" analysis, the author succeeds in giving a refreshingly new approach to a great number of results which many a numerical analyst and applied mathematician should find very useful.

Chapter I on Roots and Extremal Problems takes up iterations and fixed point theorems, for real valued functions and contractive mappings in metric spaces, and gradient methods. Chapter II, entitled Constraints, is concerned with nonlinear programming, polyhedral convex and infinite convex programming. Chapter III on Infinite Dimensional Problems is by far the longest and most substantial. It begins with the basic notions of normed spaces, Hilbert space, linear operations, differentiation, and weak convergence, and then applies these notions to a detailed discussion of the general Newton method, the solution of Fredholm integral equations, and a rendezvous problem in linear control theory. The book closes with a short section of notes and bibliographic material and a rather brief list of references. There are many exercises throughout the text, most of them fairly easy.

H. A. ANTOSIEWICZ (*Los Angeles, Calif.*)

Combinatorial methods in the theory of stochastic processes. By Lajos Tákacs. John Wiley & Sons, Inc., New York, London, Sydney, 1967. xi + 262 pp. \$12.75.

The combinatorial techniques of probability theory are based on the following simple idea. Suppose that random variables X_1, X_2, \dots, X_n have the same distribution, from which we wish to compute the probability of some event, or more generally the expectation of some function $f(X_1, X_2, \dots, X_n)$. Let G be any subgroup of the symmetric group on n elements. Then the required expectation is the same as that for the function \bar{f} defined by

$$\bar{f}(x_1, x_2, \dots, x_n) = |G|^{-1} \sum_{g \in G} f(x_{g1}, x_{g2}, \dots, x_{gn})$$

It may happen that \bar{f} is a very simple function whose expectation can readily be calculated, in which case the problem is solved.

The most important example of this method arose in the work of Sparre Andersen in fluctuation

theory, and culminated in the celebrated identity of Spitzer. In this f is a symmetric function of the partial sums $S_r = X_1 + X_2 + \dots + X_r$, and G is the whole symmetric group. This book, however, is an account of a quite different application, which also concerns the sums S_r , although its range of validity is much narrower than that of Spitzer's identity. In this G is the group of cyclic permutations, and the crucial combinatorial fact is contained in 'ballot theorems' which have been known for a century or more. The relevance of these results to the theory of stochastic processes was observed by the author some years ago, and he has since developed the idea in a long series of papers on problems in the theory of queues and dams which yield to his methods.

The title of the book should not mislead the reader into expecting an account of more than this one 'combinatorial method'. Indeed, the passing reference to Spitzer's identity completely suppresses its combinatorial nature. But as a convenient collection of the author's work on the ballot theorems and their uses, it will certainly prove a most useful work of reference.

The style is austere, results being set out in theorems with comparatively little explanation of their significance or depth. If this deters the more practical reader it will be unfortunate, since Professor Takács' methods are without doubt of potential importance in the practical application of the theory of stochastic processes. The problems he attacks are, for the most part, amenable to more classical methods, but these have the disadvantage that they present results under the heavy disguise of the Laplace transform, notoriously ill-conditioned for numerical work. The ballot theorems, in contrast, yield solutions as sums of positive terms, and these one would expect to be much more useful when numerical answers are called for. This aspect is ignored by the author—to the pure all things are pure—but one hopes that the book may stimulate others with more vulgar tastes to ask how this most elegant mathematical idea can be used in the analysis of random processes in the real world.

J. F. C. KINGMAN (*Sussex, England*)

Lambda-matrices and vibrating systems. By Peter Lancaster. Pergamon Press, Oxford, New York, London, Edinburgh, Toronto, Paris, Braunschweig, 1966. xiii + 196 pp. \$8.00.

This book introduces modern developments in the theory of finite vibrating systems from the point of view of the theory of matrices, and contains in particular many important contributions due to the author, both in the theory of lambda-matrices and in that of vibrating systems. The book is written on one hand for the pure mathematician who wants to know how his methods can be applied and on the other for the development engineer who wants to find a sound mathematical basis for the different methods he has to use in the practical work. The author does not sacrifice mathematical rigour. However, he considers it sufficient to restrict himself to the treatment of systems with linear elementary divisors only, since in most (but not all) practical applications the systems in question can be proved to have only linear elementary divisors.

After the first two chapters reminding the reader of the main results and concepts of the general theory of matrices, the author treats in 3 and 4 chapters the so-called lambda-matrices, i.e., matrices the elements of which are polynomials in λ . If the maximal degree in such an $n \times n$ matrix with respect to λ is l , the main tool of the author is the consideration of a certain *linear* pencil of matrices of the order ln , an associated matrix pencil, whose eigenvalues coincide with the latent roots of the lambda-matrix in question. Chapter 5 treats some numerical methods for lambda-matrices, in a presentation which we found particularly instructive. The next chapter discusses linear differential systems with constant coefficients in connection with the matrix theory and prepares the reader for the study of the three last chapters dedicated to general and special cases of the theory of vibration. Chapter 7 discusses differential equations of the vibration theory in connection with their physical background and in some important cases derives the solution with matrix theory methods. Chapter 8 brings a discussion of the theory of resonance testing, which is particularly emphasized by the author in view of its practical importance. We have, though, the impression that the pure mathematician will find the presentation in this chapter rather sketchy. The final chapter discusses in the main perturbation theory and develops further the well-known ideas of Lord Rayleigh.

The mathematical style of the book is rather uneven. Some things that are evident to the practical engineer are not mentioned, even if they are of importance for theoretical discussion. The theoretical part changes sometimes from pretty sophisticated arguments to rather primitive proofs which the

mathematician does not have to read and which could have been left out, with due references, of course. The author does not introduce many new terms and has therefore to use cross references, more often than the reader would like. The index is rather incomplete. We did not find in it the term *simple matrix* which is introduced in the text by the author. A large list of notations which we are accustomed to find in technical books is missing. As a matter of fact the author uses sometimes the same letter for different things. All this make the reading of the book rather hard for a not very experienced reader.

However, these are minor points. We believe that the study of this book will be rewarding to all workers in the field and will open to them many directions of useful and important research.

A. M. OSTROWSKI (*Basel, Switzerland*)

Interval analysis. By Ramon E. Moore. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966. xi + 145 pp. \$9.00.

An expert in numerical analysis may take this book with some irritated reluctance into his hands. Long ago he has had experience that the error estimates in numerical analysis are much too large and that this is often due in particular to the fact that they are usually symmetric, i.e. assign a symmetric interval around the approximation. He has already developed a lot of tricks to get around this difficulty, usually consisting in finding an interval as close as possible around the approximation in question, obtained by computing the upper and lower bounds separately. He does not feel it very necessary to make out of a collection of such tricks a Branch of Science, believing that the substance will be hardly increased through systematic treatment. And behind his resentment may lurk the unconscious apprehension that he could one day awaken and find his "Experience" reduced to triviality. Of course, a collection of tricks of the trade is very useful. Even the expert finds that there are a couple of tricks which somehow escaped his attention, and even more such which he once knew, but just forgot. Assuming that about $\frac{1}{3}$ of the substance of the book is in the principle familiar to him, he will begin looking through the book for some details which he is prepared to learn. He will, however, find that to read the book you must really *read* it, from the first page to the last. Obviously, the author was just not aware of the particular needs of the expert. Still, a more detailed index and a table of notations would certainly enhance the value of the book not only for the expert, but also for the student.

The book offers a painstakingly written account of the status of the Interval Analysis as it stands today, after some years of systematic development. In this development the author took a very prominent part, and for some years people were trying out different programs, among which those prepared by Allen Reiter have apparently been particularly useful.

The presentation begins with the discussion of the Interval Arithmetics and Analysis and the Interval Spaces with their topology, which are not at all unsophisticated. (Even the distributivity law for multiplication need not remain valid.) Then come Interval Integral (non-linear) Integral Equations, ordinary Differential Equations of n -th order, Taylor-development, which is used extensively in some of the more useful "tricks". The last chapter shows that even the Interval Analysis is not an adequate tool in the case of the topological type of the n -dimensional space. There the interval is no longer the most general neighborhood. It shows, however, that in some special cases the "Transformation Method" allows one to get reasonable results.

The proofs are often given with an unusual wealth of details; the author was obviously conscious of the fact that you cannot develop a method for $n = 2$ and say to the computer or to any programmer: "In the case of larger n proceed in the same way with obvious changes". Still, we think that to any writer in this field it would be useful to ponder over the famous example of two different proofs of the Approximation Theorem, which Littlewood presents in his "A Mathematician's Miscellany".

All workers in the field owe their gratitude to the author for having undertaken this task and carrying it through successfully, and also for having the courage to stick his neck out so far.

A. M. OSTROWSKI (*Basel, Switzerland*)

Modern nonlinear equations. By Thomas L. Saaty. McGraw-Hill Book Co., New York, St. Louis, San Francisco, Toronto, London, Sydney, 1967. xv + 473 pp. \$15.50.

This book introduces the reader to the vast field of nonlinear equations of every kind (excluding

partial differential equations), discussing basic concepts and general methods. The background of the treatment is given by Functional Analysis. Although the first chapter summarizes the essential definitions and results of Functional Analysis, a working knowledge of it ought to be acquired by the reader before studying the book. Apparently, a sufficient basis for this study is given by a preceding book (Saaty & Bram, *Nonlinear Mathematics*, 1964).

The intensified study of nonlinear methods is essentially a modern development (most of the books and papers quoted in the very rich bibliography appeared after 1955). The considerable Russian literature on the subject, both translated and original, has been extensively used by the author.

The results are often given without proofs, often they are carefully proved. We find the presentation very well balanced and agreeably readable. The "historical notes" give a kind of deepened perspective to the presentation, although a certain preference for the more modern authors makes this perspective sometimes a little lopsided, in our opinion.

The first chapter brings Basic Concepts in the Solution of Equations. It treats among other subjects topological, Hausdorff and Banach spaces, partially ordered sets, Gateaux and Fréchet differentials, inverse operators, pseudoinverses, global implicit-function theorems, fixed point theorems (Schauder-Leray, Schauder-Tichonoff, Banach, Cacciopoli, Collatz), theorems on monotonic operators by Browder, Minty and Dolph.

The second chapter, On Techniques for Nonlinear Operator Equations, treats in particular the developments of the Newton method, Picards successive approximations, generalizations of some techniques from the theory of linear algebraic equations, Ritz and Ritz-Galerkin methods.

The third chapter treats Functional Equations and Inequalities. While the part of functional equations does not bring very much additional information to that contained in Aczel's book, the last part of the chapter, which deals with functional inequalities, is particularly instructive.

The theory of nonlinear difference equations, treated in the next chapter, is relatively little known but is of particular interest today because of its importance for numerical procedures and some problems in applied mathematics. The stability situation is studied in analogy with Lyapunov's theory. A special section deals with an example of a differential-difference equation. The delay-differential equations are treated in the next chapter, which tries to cover most of the methods utilized in this field.

The next chapter, written by D. H. Hyers, treats the integral equations. The nonlinear theory by Hammerstein is carefully discussed. There is an account of the bifurcation equation and at the end of the chapter some generalizations to nonlinear operator equations are treated.

Chapter 7 is dedicated to Integrodifferential Equations, both linear and nonlinear. The vast subject is discussed mainly in a number of well chosen examples from applied mathematics. There is even mentioned a class of mixed integrodifferential-difference equations. The end of the chapter contains some indications of methodic character and such on numerical treatment. The last chapter, written by R. Syski, treats stochastic differential equations. In 110 pages a detailed account is given of the origin, application and treatment of such equations. Although the coverage is pretty complete, we looked in vain for references on the Monte Carlo methods.

As is seen from this cursory summary of the contents, there is an immense wealth of material covered. And it must be said that the coverage does not consist of a kind of annotated bibliography only, but usually provides an introduction to the subject, enabling the reader in most cases to start with the original literature as well prepared as possible.

The book is a valuable contribution to the literature in a field which becomes from year to year more important.

A. M. OSTROWSKI (*Basel, Switzerland*)

Calculus of variations and optimal control theory. By Magnus R. Hestenes. John Wiley & Sons, Inc., New York, London, Sydney, 1966. xii + 405 pp. \$12.95.

An account is presented of conditions for a minimum in a wide class of problems. The choice of material and style invite comparison with the standard reference *Lectures on the Calculus of Variations* by G. A. Bliss. The reviewer found Hestenes' book distinctly the more satisfying of the two. An important advantage is that the necessary conditions are derived in a form immediately applicable to problems with inequality constraints which occur in optimal control theory and elsewhere.

The first chapter provides necessary background on convexity and from advanced calculus. A

Lagrange multiplier rule is proved for minima of functions of several variables subject to inequality constraints. Then comes a rather complete treatment of necessary and sufficient conditions for the classical fixed endpoint problem, without side conditions. To treat problems with side conditions in the form of differential equations, differential inequalities, etc., a very general multiplier rule is then proved. It applies to the problem of minimizing a function $J_0(x)$ on some linear space subject to a finite number p of relations of the form $J_\gamma(x) = 0$ or $J_\gamma(x) \leq 0$. An important idea in the proof is that of derived cone for a subset of $(p + 1)$ -dimensional Euclidean space. The connections with the Hamilton-Jacobi equation and dynamic programming are established through the concepts of optimal field and program. The Mayer problem is discussed from a geometric viewpoint. The final chapter considers some problems with bounded state variables. Throughout the book the theorems are illustrated with examples and exercises.

WENDELL H. FLEMING (*Providence, R.I.*)

Topological spaces. By Eduard Čech. Revised edition by Zdeněk Frolík and Miroslav Katětov. Interscience Publishers (A Division of John Wiley & Sons), London, New York, Sydney, 1966. 893 pp. \$22.50.

The Czech edition of "Topological Spaces" by Eduard Čech was published in 1959. The present English edition, with substantial enlargement and rearrangement, is a complete revision written by Frolík and Katětov after the death of Čech.

Among the large number of existing books on general topology, this book stands out with unique features which represent strongly and consistently the specific point of view of the authors.

As a rule, all topics are studied in a setting as general as possible, and the approach is unusually systematic. The emphasis is on general spaces. Thus the treatment of compactness and completeness is relatively brief, and most topics on connectedness are omitted. Along with topological spaces, uniform and proximity spaces and their interrelations are thoroughly studied. Another feature is a detailed exposition on an axiomatic, but not strictly formal, development of the set theory, including an introduction to the theory of categories. Partly due to the extremely systematic approach of the book, the terminology and notation deviate sometimes from the common usage, and quite a number of new terms have been introduced.

The book is divided into seven chapters and an appendix (pp. 779-819) on compactness and completeness. The titles of the successive chapters are: classes and relations (pp. 17-90), algebraic structures and order (pp. 91-231), topological spaces (pp. 233-394), uniform and proximity spaces (pp. 395-476), separation (pp. 477-549), generation of topological spaces (pp. 551-677), generation of uniform and proximity spaces (pp. 679-778). The first two chapters are written by Katětov, the other chapters and the appendix by Frolík. The exercises are placed at the end of the book (pp. 821-869) and grouped by the sections. To avoid biased personal judgements, the authors decided to make no historical remarks and to give no list of bibliography.

The book is not intended as a textbook. It is a very well written scholarly work which will undoubtedly be an important reference for the mathematicians interested in general topology from a very general viewpoint.

KY FAN (*Santa Barbara, Calif.*)

Automata theory. Edited by E. R. Caianiello. Academic Press, New York, London, 1966. xiv + 342 pp. \$14.00.

The twenty-seven articles of this book (eighteen in English, nine in French) constitute the proceedings of the sixth session of the International School of Physics, which was held in Ravello, Italy, in June 1964. This session was dedicated to automata theory because (quoting from the editor's preface) "from the point of view of electronic computation, automata theory has offered the means for studying the functions and potentialities of phenomena *in depth*; from a crude piece of engineering, the computer becomes an organism whose organizational principles can be discovered and put to useful purpose, much in the same way as physiology explains anatomy. There is, in fact, a growing tendency toward the study of automata whose behaviors may come near to those of living organisms..." The title of the book is deceiving, however, since the collection of papers does not present a unified account of

any theory. (As a matter of fact, automata theory is simply *not* a unified theory at present in the sense that group theory, or the theory of complex variables, is a theory.) Furthermore, several large problem areas usually classified under automata theory are omitted, which is also a criticism of the title and not of the volume.

In the reviewer's opinion, the volume on the whole is more useful to someone in the field than someone outside who wants to know what automata theory is all about. Yet many articles are as readable to the outsider as to the insider, considering both motivation and prerequisite background. Thus, for example, most readers could come away from C. Berge's paper with a feeling for the relevance of graph theory to the theory of codes. Similarly, no special knowledge (beyond some elementary logic) is necessary in order to read the exposition by C. Böhm and W. Gross of lambda calculus and combinatory logic, especially if the reader is interested in programming languages in the abstract. R. Büchi explains the basic concepts of finite automata in an algebraic framework, so that exposure to modern algebra is all that the reader needs. Also useful to readers outside the automata field are: the second article by A. Caracciolo di Forino, which is about generalized Markov algorithms, with some mention of Turing machines and formal languages; J. D. Cowan's article on the various techniques of building statistically reliable computing machines (etc.) out of statistically unreliable elements; M. Davis's second article, an eleven-page introduction to recursive functions; two articles on the animal nervous system, one by E. M. Harth and the other by W. S. McCulloch and W. L. Kilmer; L. Nolin's approach to Turing machines and computer programming, which involves a sort of flow-chart analysis of an algorithm.

In the reviewer's opinion, the most interesting article from the point of view of applied mathematics is M. P. Schützenberger's first article. The cardinal principle is that if V is a pseudo-variety of groups then the family of finite monoids (i.e. semigroups with identity) whose subgroups belong to V is also a pseudo-variety. This cardinal principle has been used to solve a number of problems in automata theory, as shown by another article by Schützenberger and an article by L. A. M. Verbeek. (One of these problems solved by Schützenberger was posed by the reviewer, which explains his enthusiasm.)

Several expository articles are most useful to those already knowledgeable in the field. An article by M. Arbib tells about the work of Blum, Rabin, and Gödel in the speed-up of computations of recursive functions and proofs. The first article by M. Davis tells about research by himself and others on Hilbert's tenth problem, i.e. on whether or not there is an algorithm that determines when a polynomial diophantine equation has a solution in integers, and related problems. H. Korezlioglu gives an exposition of threshold logic in his paper, "Graphes de Transfert des Réseaux Neuroniques." A. Gill presents an exposition of linear finite automata, along with an algorithm for reducing such automata, in the first of his two articles; in the second (which is a research paper) he investigates such automata that have no inputs. M. Nivat presents a self-contained but mathematically precise exposition of the algebraic theory of codes; he explains the quotient monoid of a code, complete codes, decoding by a finite automaton, codes with the finite-delay property, prefix codes, composition of codes, and synchronizing codes. M. Rabin's article presents the fundamentals of probabilistic automata.

Many of the articles of the volume are research papers, including some of the papers already mentioned and all of those now to be described. L. Amar and G. Putzolu try to generalize on Kleene's regular expressions by introducing a new operator on sets of words; in their short article they get only as far as a very small subclass of the class of context-free languages. M. Borillo investigates algorithms for finding the shortest path through a graph, to determine which algorithm is optimal. P. Camion seeks to generalize results of Berge on the borderline of graph theory and linear algebra. The first article by A. Carrocciolo di Forino discusses many-valued logics and offers an equational characterization of a selection function. (f is an n -ary selection function if, for any given σ , there is a positive integer $i \leq n$ such that $f(\sigma, x_1, \dots, x_n) = x_i$ for all x_1, \dots, x_n . The reviewer finds, however, that his set of equations is not a characterization, and, by a simple model-theoretic argument, that there is no equational characterization.) M. Gross discusses linear context-free grammars, defines a new operator on linear grammatical rules which take him outside the class of context-free languages, and then discusses applications to natural language. J. Holland seeks to put forth a universal scheme for studying computation and adaptation (in the sense of control mechanisms). L. Löfgren investigates the information content of the proposition that says an object is a member of a set; the reviewer, however, finds the problem uninteresting when the author takes the study beyond the finite sets. J. Larisse and M. Schützenberger present some results on non-homogeneous Markov chains.

All in all, the volume is a rather useful one.

R. McNAUGHTON (Troy, N.Y.)