POTENTIAL FLOW WHEN A UNIFORM STREAM OF INVISCID LIQUID IS DISTURBED BY AN OVAL OF CASSINI

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Consider two fixed points A, B where \( AB = 2c \). The locus of a point \( P \) which moves so that \( AP \cdot BP = b^2 \), a positive constant, is a curve called an oval of Cassini. When \( b \gg c \), the curve is a closed convex oval. As \( b \) decreases the curve ceases to be convex, developing a "waist", and finally, when \( b = c \), becomes a figure of 8 known as Bernoulli's Lemniscate. If \( b < c \) the locus splits into two closed curves, one within each loop of the lemniscate, and finally degenerates into the two points A and B when \( b = 0 \). We shall be concerned here only with the case \( b \geq c \). Taking A and B to be the points \( z = -c \) and \( z = c \), \( z = x + iy \), these curves are given by \( \xi = \alpha \), where \( \alpha \) is a constant parameter, in the net

\[
z = c(1 + e^{2\xi})^{1/2}, \quad \xi = \xi + i\eta, \quad \alpha \geq 0,
\]

the lemniscate corresponding to \( \alpha = 0 \).

In two-dimensional flow let one of these curves, say \( \xi = \alpha \), disturb the uniform stream whose complex potential is \( Ue^{-i\theta}z \). The fluid then occupies the region for which \( \xi > \alpha \). Therefore of two points for which the real part of \( \xi \) is \( \xi \), or \( 2\alpha - \xi \), only one lies in the region of the flow.

From (1) we have

\[
Ue^{-i\theta}z = Ue^{-i\theta}c(1 + e^{2\xi})^{1/2} = F_1(\xi) + F_2(\xi)
\]

where, for sufficiently great values of \( \xi \), we have by the binomial theorem

\[
F_1(\xi) = Ue^{-i\theta}ce^\xi, \quad F_2(\xi) = Ue^{-i\theta}c(\frac{1}{2}e^{-\xi} - \frac{1}{4}e^{-2\xi} + \cdots).
\]

Thus \( F_2(\xi) \to 0 \) as \( \xi \to \infty \) and the complex potential of the given stream tends to \( F_1(\xi) \). Now since \( \overline{f(\xi)} = f(\overline{\xi}) \), where the bar denotes the complex conjugate, we see that \( \overline{F_1(2\alpha - \xi)} = Ue^{i\theta}ce^{2\alpha - \xi} \) tends to zero when \( \xi \to \infty \). Thus by a general method [1] the complex potential for the disturbed flow is

\[
w = F_1(\xi) + \overline{F_1(2\alpha - \xi)} = 2Uce^\alpha \cosh (\xi - \alpha - i\beta)
\]

The verification is immediate, for when \( \xi = \alpha \), i.e. on the oval, \( w \) is real so that the stream function is zero and the oval is a streamline, while when \( \xi \to \infty \), \( w \) tends to \( F_1(\xi) \) which is the complex potential of the stream.

We also notice that the only singularity of \( F_1(\xi) \) is at infinity in the flow and therefore \( \overline{F_1(2\alpha - \xi)} \) introduces no new singularities into the flow.

The case of the lemniscate, \( \alpha = 0 \), is interesting, for this curve has a double point at the origin, where the two tangents intersect at right angles. Thus the streamline

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\[ \psi = 0 \] intersects itself at right angles at the origin which is therefore a stagnation point whatever the value of \( \beta \), the direction of the stream.

REFERENCE