QUARTERLY OF APPLIED MATHEMATICS

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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors’ cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for “author’s corrections.”

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing penciled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter t and the prime (‘), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set above letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which follow the letter should be used.

Square roots should be written with the exponent ½ rather than with the sign √. Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponents with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus, exp \((a^2 + b^2)^{1/2}\) is preferable to \(e^{(a^2 + b^2)^{1/2}}\).

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidsus. Thus,

\[
\frac{\cos \left(\frac{xx}{2b}\right)}{\cos \left(\frac{xx}{2b}\right)} \text{ is preferable to } \frac{\cos \frac{x}{2b}}{\cos \frac{x}{2b}}.
\]

In many instances the use of negative exponents permits saving of space. Thus,

\[
\int u^{-2} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.
\]

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

\[(a + bz) \cos t \text{ is preferable to } \cos t (a + bz).\]

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

\[
[(a + (b + cx)^n) \cos ky]^{1/2} \text{ is preferable to } (a + (b + cx)^n) \cos ky^{1/2}.
\]

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

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Authors’ initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, On the Flow of Viscous Fluids is preferable to On the Flow of Viscous Fluids, but the corresponding German title would have to be rendered as Über die Strömung zäher Flüssigkeiten.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., chap., p., etc.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, “Eq. (25)” is acceptable, but not “the preceding Eq.” Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus “boundary conditions” should always be spelled out and not abbreviated as “b.c.,” even if this special abbreviation is defined somewhere in the text.
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Book Review Section


Time series analysis has a long history but it was not established as an independent branch of statistical theory until some 30 or 40 years ago, when it became clear that the analysis should be based on the explicit specification of probabilistic models. This made it possible to discuss suggested methods of analysis in an objective manner; to look for optimal solutions, etc. One soon turned to the more sophisticated models of stochastic processes and, particularly, the stationary case received a great deal of attention. Today, we have access to a large and varied collection of analytical techniques and the statistical theory has reached a certain degree of maturity.

At the same time as the theory was developed it was used by research workers in other fields, in the beginning especially in communication engineering, but later on the applications have spread to many other areas. One of the main obstacles has been the large amount of computation that time series methods often require. When the digital computers as well as convenient programming languages became generally available, the situation looked more hopeful. In recent years we have seen the development of efficient algorithms intended for time series analysis.

The present book deals with the computational aspect of time series analysis. It contains a detailed study of the numerical problems typical for this subject. To give some idea of what topics are treated, let us mention the fast Fourier transform, digital filtering and prediction of stationary series, and detection problems. An interesting and somewhat unusual feature of the book, apparent already in its title, is the emphasis given to multiple series. This is a great help since very often the analyst is confronted with multiple series and the additional difficulties. The approach is based on spectral theory and it is linear-quadratic with a few exceptions. Practically every algorithm discussed by the author is accompanied by its Fortran program. This set of programs is one of the most valuable features of the book.

Chapter 3 is a bit different from the rest of the book in that it deals with questions related to wave propagation in heterogeneous media. It provides an illustration of how to use the techniques treated elsewhere in the book.

This reviewer believes that we shall see a shift in emphasis, not only as far as time series analysis is concerned, but for statistics in general, to the computational aspect. This attractive book can be seen as an indication of the possible consequences for statistics when the full potential of the digital computer is realized. To quote the author:

"The digital computer has eliminated the need for the middle-men who have always stood between the source of information as represented by time series and the ultimate consequences of this information."

The style of the book is clear, it is precise without being pedantic, and it reads well most of the time. In the last two chapters it is occasionally less easy to follow the author's reasoning. This may be just because the multidimensional case is essentially more difficult to present. We should be grateful to Dr. Robinson for presenting us this useful book.

This reviewer takes exception to the statement on p. 179 concerning the role of the computer in time series analysis: "The true value of this new aspect of time series analysis cannot be underestimated."

Ulf Grenander (Providence, R. I.)


This book summarizes the work of the last 8 or 10 years on random matrices and presents its results in a coherent fashion. It may serve as a substitute for the collection of original papers, edited by Charles E. Porter and published about 2 years earlier, also by the Academic Press. However, the introductory
review to this earlier book, written by Porter, remains rewarding reading even now. On the other hand, Mehta's book contains a good many new results, due principally to himself, which were, naturally, not anticipated in Porter's collection. It also gives a more accurate picture of older work than Porter's collection of new articles could give.

The motivation for the physicist's interest in random matrices stems from quantum mechanics' interpretation of the energy levels as characteristic values of a self adjoint operator. As the author explains in the first chapter, these characteristic values have a lower bound and the characteristic values close to this bound can be interpreted quite well on the basis of approximate solutions of the characteristic value problem. On the other hand, the characteristic values well removed from the lower bound are, even within unit energy interval, so numerous that their exact calculation is a hopeless task. It would be also of little interest. Instead, one is interested in their average properties, such as their density (i.e. their number per unit energy interval), the statistical distribution of the spacing between neighbors, the correlation between adjacent spacings, and similar quantities, and the book is devoted to the calculation of these. The model used is, practically throughout the book, the so-called Wishart ensemble or variations thereof; the Wishart ensemble is an ensemble of matrices, as a rule symmetric or hermitean, without any correlation between the matrix elements unless demanded by the symmetry condition, each matrix element showing the same Gaussian distribution with zero mean value. It is, naturally, an open question whether this ensemble is an apt substitute for the self adjoint operator which according to physical theory gives the energy levels, and this question is little touched upon in the literature. It is certain that the mathematical investigation of the ensemble yielded many results which are interesting in their own right.

Actually, the original incentive for the investigation of the ensemble is given, except for the first chapter, very little attention in the book. Rather, the interest soon turns to mathematical problems as such and the book exhibits surprising manipulative skill throughout. This reader, at least, marvelled at the wealth of clever ideas used in the calculations, the far-reaching and concrete nature of the results. These provide an answer to all the questions mentioned above and to a multitude of others. The principal originators of the calculations reviewed are M. Gaudin in France, F. Dyson in this country, and last but not least, the author, Mehta, himself. For further names, see Mehta's list of references. The mathematical methods employed may not be very modern or very sophisticated, but they are ingenious and the reading of the book gave this reviewer a great deal of pleasure.

No review of a book is complete without some critical remarks. These concern, in the present case, a moderate amount of carelessness in some of the explanations. This is particularly true of the first chapter. Thus, the problem of Hardy-Ramanujan, discussed on page 5, is incorrectly formulated: it does not deal with an arbitrary decomposition of an integer \( n \) into a sum of positive integers (there are \( 2^{n-1} \) such decompositions) but into a sum in which no member exceeds the preceding one. The number of such decompositions is difficult to estimate and is the number for which Hardy and Ramanujan gave an asymptotic expression. The \( x \) on page 6 is not the reduced width of an energy level, but this width divided by the average reduced width. The number of electrons in a metallic particle of size \( 10^{-6} \) to \( 10^{-7} \) cm is not \( 10^4 \) to \( 10^5 \) but can be as low as 200, and at a size of \( 10^{-7} \) cm rarely exceeds 3000. None of these errors affects the main line of the story—this reviewer did not notice any truly significant error in the book—but it would have been better to avoid them, nevertheless.

The book is recommended without reservations to those theoretical physicists or mathematicians who are interested in the problems outlined and also to those who find pleasure in learning about old-fashioned, skillful mathematical operations.

E. P. Wigner (Princeton, N. J.)

_Spektraldarstellung linearer transformationen des Hilbertschen raumes._ By Béla Sz.-Nagy.


This book is essentially a reprint of the first edition published in 1942. It derives the most important theorem of Hilbert space theory, namely the spectral representation theorem for normal operators. (Normal operators are operators which commute with their adjoints, the two most important classes of normal operators being the unitary and the self-adjoint transformations.)

Not much more than the idea of a vector space is assumed known, Hilbert space is defined and all the elementary results needed (such as the triangle inequality) are proved, so that the book is practically
self-contained. A simple and very direct proof of the spectral representation theorem is given for bounded, self-adjoint operators by splitting $A_\lambda = A - \lambda I$ into positive and negative parts

$$A_\lambda = A_\lambda^+ - A_\lambda^-$$

where

$$A_\lambda^+ = \frac{1}{2}(A_\lambda + \sqrt{A_\lambda^2}),$$
$$A_\lambda^- = \frac{1}{2}(-A_\lambda + \sqrt{A_\lambda^2}).$$

The projection operator $E(\lambda)$ on the null-space of $A_\lambda^+$ gives the desired resolution of the identity. A proof for unbounded normal operators is obtained by considering the bounded self-adjoint operator $(I + A^*A)^{-1}$. The original proof of von Neumann's for self-adjoint operators is also given.

Brief accounts of some additional topics (such as perturbation theory, semi-groups of normal and self-adjoint operators) are given.

For further developments one must refer to other works published in the last few years—for example, to Ahiezer and Glazman: Theory of linear operators in hilbert space (Russian) 2nd edition, Moscow, 1966, for applications to ordinary differential equations and to K. Maurin: methods of hilbert space, Warsaw, 1967, for applications to partial differential equations.

The theory covered in Sz.-Nagy's book had reached essentially final form twenty-five years ago. His account of it is remarkably readable, perhaps the best ever written.

R. C. T. SMITH (Armidale, N. S. W.)


This book is a clearly written, essentially self-contained presentation of some aspects of the analytical theory of periodic solutions of nonlinear autonomous systems of ordinary differential equations. The author treats in detail the problem of the existence and stability of periodic solutions, as well as the numerical computation of such solutions. He discusses the effect of perturbations in the differential equations on the oscillatory properties of the solutions. There is a chapter devoted to the center problem in high dimensions as well as one concerned with the determination of those second order nonlinear oscillators which have periodic solutions with periods specified a priori as a function of the amplitude. The above topics are presented in a unified manner by the introduction of a moving orthonormal system along orbits. The exposition is made self-contained by the inclusion of chapters on the basic elements of vectors, matrices, general differential equations, the Newton and step-by-step methods for numerical solutions. The scientist interested in nonlinear oscillations will find much valuable information here which appears in book form for the first time.

Jack K. Hale (Providence, R. I.)

Stationary and related stochastic processes—sample function properties and their applications.


The total number of books on the theory of stochastic processes has, by now, exceeded several dozen. This number is being increased by one with the publication of the present book. However, this one unit is of very high intellectual specific gravity: Cramér's and Leadbetter's book is not a rehash of its predecessors, but represents an extremely valuable addition to the existing literature on stochastic processes and fills an important gap in this literature.

To this day, and even in the most detailed monographs on stochastic processes written for "pure" mathematicians (such as the classic book by J. L. Doob entitled Stochastic Processes, Wiley, 1953) there is no mention of the wide area of "zeros", "level crossings", "extrema" and other essential characteristics of stochastic processes, which play such important parts in many applications. Equally, no mention can be found of such concepts as envelope or instantaneous frequency which are also often
employed in applications. At the same time, books written for engineers (such as *Principles and applications theory of random noise theory*, J. S. Bendat, Wiley, 1958 or *Introduction to statistical communication theory*, by D. Middleton, McGraw-Hill, 1960) devote much space to the study of the level-crossings and related topics and utilize the concepts of envelope and instantaneous frequency (in both cases, the most important references are to the pioneering work of S. O. Rice). However, in such books, the presentation is held on an "engineering" level of rigor and, consequently, suffers from spurious limitations, from flaws in the details of proofs or even in the basic formulation of the results. Finally, an entirely different place is occupied by the problems of such local properties of sample functions (i.e., trajectories) of stochastic processes as continuity, differentiability, etc. These problems are treated only cursorily and incompletely in books on stochastic processes and are ignored by the authors of all books intended for engineers.

The fundamental importance of Cramér's and Leadbetter's book turns precisely on the fact that they do provide us with a mathematically rigorous presentation of level-crossing problems and the theorems on local properties of trajectories of stochastic processes; one, moreover, which is as simple as possible and which does not demand that the reader should possess unusual mathematical abilities. In addition, they illustrate their results with many examples of engineering importance. The book addresses itself principally to mathematicians who have an interest in the applications of the theory of stochastic processes to engineering problems. It is also suitable for engineers who are not content with heuristic concepts and who feel the need for rigorous definition and proofs. Both categories of readers will find the book interesting and useful.

Chapter 1 of the book introduces heuristically the concept of a random variable and of a random (stochastic) process. Chapter 2 contains a rigorous introduction to the fundamental concepts of the theory of probability and indicates to the reader the relevant text-books. Chapter 3 presents some fundamental results of the theory of stochastic processes. In doing this, the authors cannot, evidently, manage without measure theory, but they limit themselves to the basic minimum of information relevant for the purpose. Taking into account the interests of readers who are engineers, they endeavor to explain the essence of the concepts and theorems treated in as simple a way as possible. The same chapter presents the general properties of stationary streams of events which are very important for future consideration. In this presentation the authors follow the example of, principally, A. Ya. Khinchin and other Russian authors. Chapter 4 discusses the fundamental theorems on the sufficient conditions for continuity, differentiability, etc. of almost all the trajectories of a general stochastic process \( \xi(t) \) (without making the supplementary conditions for the normality and stationarity of \( \xi(t) \)). In their endeavor to keep the exposition simple, the authors formulate these theorems in a manner to dispense with the traditional but difficult concept of a separable stochastic process. In the case of several related theorems, the authors frequently omit to reproduce the full proofs of all of them but confine themselves to short explanations or even, to the mere formulations of the result, referring the reader to research papers in which such proofs can be found. For example, the condition for the absence of discontinuities of the second kind in the trajectories of the process are treated in this manner. The same approach is also often used in later chapters. Chapter 5 is devoted to a study of stochastic processes with finite second-order moments. It is here that the authors discuss the general theorem on the conditions for continuity, differentiability and integrability in mean squares sense and, incidentally, introduce the conditions of continuity, differentiability and integrability with probability one in terms of second moments. The short Chapter 6 discusses processes with orthogonal increments and stochastic integrals. Chapter 7 and 8 contain an exposition of the basic facts of spectral theory of stationary stochastic processes (including theorems on the local properties of sample functions as well as the ergodic theorem of Bochner-Khinchin) together with their most important generalizations. Chapter 9 studies the local properties of trajectories of Gaussian (normal) processes. In doing this, the authors do not attempt to prove the strongest existing theorem on the conditions for continuity, differentiability, etc., but limit themselves to proofs of weaker assertions which follow from the general conditions given in Chapter 4. These, however, are amply sufficient for all applications. In the case of subtler theorems, or theorems whose proofs are more complicated, they are satisfied with formulations and references to the literature.

The four succeeding chapters are wholly devoted to the study of level-crossings and related problems. Chapters 10 and 11 rigorously, and under quite general conditions, derive formulae for mean values, dispersion and factorial moments of higher orders of the number of crossings of a given level of stationary stochastic processes. The particular case of a Gaussian process is treated in great detail. Furthermore, the chapter discusses certain results concerning the intervals between two neighboring level crossings, characteristic excursions above a level and the level crossings of the envelope of a sta-
tionary Gaussian process. In Chapter 12, the authors provide us with a detailed proof of the important limit theorem which is very often formulated incorrectly in engineering text-books. According to this theorem, the u-crossings of a stationary Gaussian process \( \xi(t) \) tend, as \( u \to \infty \), under some general conditions, to a Poisson process. This is the most complicated proof in the book. The same chapter introduces the results on the probability distribution for the maximum of a stationary Gaussian process and for the length of an excursion above a high level. Chapter 13 generalizes earlier results to include the case of intersection of an arbitrary curve \( u = u(t) \) with a Gaussian non-stationary process. Finally, the last two chapters (14 and 15) are of a somewhat more specialized character. They discuss some applied problems connected, in particular, with envelopes and instantaneous frequencies, and require for their solution the application of the theoretical results of the earlier chapters.

The method of exposition employed in Cramér's and Leadbetter's book is as distinguished as in the earlier books of the senior author which, deservedly, became so popular. The book is methodical but also simple and lively. It appears to me, however, that the style of the last two chapters which deal with concrete engineering problems does not quite fit into that of the rest of the book. I can also note that, on p. 147, it would have been useful to display, without proof, the more general conditions of convergence with probability one of the time average of a stationary (wide sense) process to the mathematical expectation. This can be found in a paper by Verbitskaya to which the authors refer. It is, equally, a pity that the authors were unable to indicate the possibility of strengthening some of the important results of Chapters 12 and 13 which are described in the papers of Yu. K. Belayev (Teoriia veroiatnosti i ieo primenemiia, 12, 444, 1967) and M. R. Leadbetter (Bull. Amer. Math. Soc. 73, 129, 1967), because they were published later than the book under review.

The book has been produced magnificently. However, your reviewer could not understand the purpose of the flamboyant design on the dust-jacket and suspects that it has nothing to do with the contents of the book.

A. M. Yaglom (Moscow, U.S.S.R.)


This is a very useful book dealing with the important and at times complex concepts clustered under the heading of "order" in vector spaces. The author writes clearly, the proofs are complete, there are many examples at various levels of difficulty, and there are a bibliography and some historical notes which seem to be well-balanced. Added to all this, the book has the virtue of being short. The first chapter develops the general theory of ordered vector spaces. Here one finds all the preliminary bride's dowry of basic concepts which one must carry through to the remainder of the book. Then, there are discussions of cones, bases, simplexes, lattice ideals, bands, order convergences (of nets) and the important theorems on these are proved. The second chapter considers topological vector spaces. A principal consideration here is that of determining conditions for the extendability to the entire space of positive mappings defined on a subspace and also conditions to ensure the continuity of positive mappings. Also considered are the implications of the lattice axioms on the order structure of the space of continuous linear functionals and on the order continuity of continuous linear mappings. Chapter Three is devoted to the study of two principal topologies attached to an ordered vector space: the order topology and the so-called \( \mathcal{O} \)-topology. These are extreme topologies which are compatible with the lattice structure. The last chapter is entitled "Selected topics in the theory of ordered topological vector spaces." Considered are: relations between order completeness and topological completeness; topological properties of order convergence; order properties of spaces of continuous linear mappings. Finally there is a nice fourteen-page appendix summarizing the theory of locally convex spaces. Let no one think that all this material is to be absorbed in the languid manner of a post-card bear licking wild honey. The arguments are pedagogically ordered (\( \leq \)) and the chains are not short; there is considerable room for heaving and panting. This is probably due to the fact that the subject matter of ordered vector spaces is not as yet in highly polished form. However, it is safe to say that a very few years from now second-year graduate students, with their incredible appetite for learning to live on the frontier, will consider it all very straightforward, probably because they will have had a book like this one at their disposal to guide them in their first steps.

E. R. Lorch (New York)

A frequent problem in automatic control theory is to make some estimate of a parameter of the control system (based on noise corrupted observations), and then to select a control or gain constant which is optimum in some prescribed statistical sense. This book (of 211 pages) is devoted to an exposition of the methods of statistical decision theory, and to some of its applications to problems of the above type in control theory. The authors present a readable discussion of the standard methods of statistical decision and hypothesis testing theory; minimizing the Bayes risk, Neyman-Pearson method, sequential and non-sequential methods. The volume is apparently intended for an audience of control engineers and the vocabulary has been adapted, in part, to the vocabulary of the potential users. This certainly seems to be a worthwhile idea.

In the case of linear systems a typical problem concerns the estimation of a parameter of the control system. The parameter plus corrupting noise is observed, some prior distribution for the parameter is given, and one must select a 'gain' (the decision) so as to minimize some risk (which is determined in a natural way by the performance of the control system). The problem is investigated also from the point of view of the Neyman-Pearson test, and a number of variations are discussed. Although this type of problem is of interest it is not always the most natural. For example, for the scalar system \( x = bx + ku \), where \( b \) is unknown, the value of \( b \) must usually be inferred from observations of the type \( x(t) \) + noise, rather than from \( b + \) noise. Although the methods are similar in principle, the computational problems associated with the former type of observation are vastly greater than those associated with the second type. There is an indirect approach which does yield the minimum Bayes risk formulation of the author—even if \( x(t) \) + noise type observations are taken: Let the parameter take values \( \theta_1, \ldots, \theta_n \); let \( x(t) + \) noise be the observations; use a maximum likelihood technique to compute the most likely parameter; the technique gives a sequence \( p_{ij} = \text{Prob} \{ \theta_i \text{ selected when } \theta_j \} \); these \( p_{ij} \) are used in the minimum Bayes risk formulation to determine the best gain (decision). Also the problems of dynamic optimization are not considered; generally, the 'gains' (\( k \)) are selected after all the observations have been taken.

The next to last chapter (40 pp) discusses a problem in the estimation of a parameter of a non-linear feedback system—the system contains a relay and an unknown time constant. Owing to the non-linearity, the computations involved in the decision procedures are rather hard and so the authors introduce the concept of 'statistical linearization' which, in effect, allows the replacement of the non-linearity by a linear term which is optimum in a given sense. This vastly simplifies the requisite computations, although it is difficult in general to determine the exact difference between (say) the risk computed in this way from the true risk for the original problem. Presumably they are close in cases where 'statistical linearization' is effective, and the general idea is probably worthy in at least these cases.

H. Kushner (Providence, R. I.)


This book presents a well-assorted collection of elementary examples illustrating the use of simple mathematical tools in economic theory and management science. The twelve topics are interesting in themselves, and they are neatly developed with an emphasis on ideas rather than techniques, making this a little masterpiece of pedagogy. The mathematician who has not encountered these applications before in textbooks on Markov-chains, difference equations, dynamic programming, game theory and inventory theory will find this informative and often amusing reading. This is excellent supplementary material for introductory mathematics courses for economists and social scientists.

M. J. Beckmann (Providence, R. I.)

This is the second edition of the book originally published in 1957. Since then both the theory and the computational art of difference approximation for initial value problems has advanced very much, and the book has been revised considerably. As before the book is divided into two sections. The first part treats the theory and its size has been doubled. It gives a rather complete account of the stability theory for the Cauchy problem and provides techniques for constructing stable approximations. The second part is concerned with applications and here it is the section on fluid dynamics which has been rewritten and enlarged from 40 to 100 pages.

In my opinion the book is very well written and gives a very good account of the present state of the theory and methods which are used in practice. In particular I like the combination of both theory and applications and the wealth of unsolved problems.

Heinz O. Kreiss (Uppsala)


If contemporary economic theory is dominated by the economies of growth, contemporary mathematical economics may be said to be fascinated with optimal economic growth. The present collection of research papers gives an excellent presentation of the state of the art. The first essay by Karl Shell, the editor, takes the reader into the very heart of the matter. Actually the problem of optimal capital accumulation—or optimal saving—was treated as an exercise in the calculus of variations in the twenties by Frank Ramsey, a protegé of J. N. Keynes, who is perhaps best known for his work on the foundations of mathematics. If the criterion function is the—undiscounted—integral of consumption, the solution turns out, perhaps not surprising, to be bang-bang: save until the “golden rule for a golden age” applies: the interest rate equals the growth rate or equivalently all nonlabor income is saved. This result is proved by means of the maximum principle of Pontryagin first for a one-sector economy without technical progress. The result is also a generalization of the famous turnpike principle: the optimal, path for attaining a distant goal leads through a process of balanced growth most of the time. In extending the analysis to the case of exogenous technical progress, difficulties are encountered such as multiple switches and, in fact, no maximum need exist. The turnpike property may be extended to a two-sector economy producing a single consumption good and a single capital good, when production of the capital good is more labor intensive than that of the consumption good. In the converse case the economy should specialize first in the production of the capital good and only as the end of the planning period is approached will there be any consumption.

It is indicative of the present state of growth theory that a great number of apparently quite special cases have to be examined after the general lines of the inquiry have been laid down. These include analyses of the optimal rates and direction of technical change; of the cases where growth of the level of technology depends itself on the level of commodity output; the case of an open economy of “moderate size,” in particular the effects of foreign borrowing; the allocation of investment and transportation in a two-region economy, and the case of an unlimited supply of labor; the effects of evaluating leisure along with consumption; and the case of a two-sector model with fixed coefficients, previously analyzed by Gale. The basic model is also compared and integrated with the standard neoclassical growth models reflecting actual rather than optimal behavior, among others in a brilliant essay by Samuelson.

Needless to say, a summary review like this cannot begin to do justice to the wealth of material and its often penetrating analysis. A rich mine holding the promise of still greater riches yet to be unlocked has been opened up here. The seminar of which this is the printed result may very well be compared to the famous colloquium in Vienna which produced papers by Menger, Wald, Von Neumann and others, the effects of which extend well into the present. For a mathematician interested in modern mathematical economics there is no shorter path than the study of this volume.

M. J. Beckmann (Providence, R. I.)

This book gives a remarkably concise and complete exposition of the theory of estimating simultaneous equations, the central problem in modern econometrics. The student is expected to have a solid background in elementary statistical theory including some multi-variate analysis. The main part of the book develops the theory of parameter estimation for the standard linear model in its non-temporal version. A subsequent chapter adds a briefer treatment of the problems of lagged endogenous variables and of autocorrelated disturbances. Prior to the problem of estimation is that of identification—which coefficients can be estimated given the assumption that certain variables are excluded from certain equations. The full-information and limited-information variants of the maximum likelihood approach are presented in full detail. There follows an exposition of $K$ class estimators, including the well-known two-stage least squares method and some more recent variants thereof. A special chapter is devoted to measures of correlations and to some tests. A numerical example involving five equations is worked out to illustrate all methods.

Within its limitations of space the book achieves its purpose admirably by sticking to the essentials: the statistical theory. Problems of a computational nature and small sample aspects that at this stage of our knowledge can only be treated by simulation experiment are not discussed. A comprehensive bibliography concludes the book. Foreign language contributions, however, even those of Malinvaud, are missing.

This book is a welcome addition to the growing literature of econometric texts, particularly useful to those who are already familiar with statistical theory and to those looking for a rapid survey of the field.

M. J. Beckmann (Providence, R. I.)