ESTIMATIONS OF THE LENGTHS AND PERIODS OF CLOSED TRAJECTORIES*

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Let Γ be a closed trajectory, of period ω, of the two-dimensional autonomous system of differential equations

$$dx/dt = P(x, y), \qquad dy/dt = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ belong to class C^1 for all x and y. Let B be a region of finite area $|B|$, free of singular points and containing in its interior the region A, in which lies the closed trajectory Γ. Under these assumptions S. P. Diliberto [1] has given bounds for ω. This bound depends solely on norm $||X||$ of the vector-function $X(x, y) \equiv \{P(x, y), Q(x, y)\}$, curvature $k(x, y)$ of the trajectory of system (1) passing through (x, y), curvature $h(x, y)$ of the orthogonal trajectory and some measure of the boundness properties of A.

In a recent paper P. J. Lau [2] deals with the same problem. By strengthening the suppositions on the boundaries of A he is able to define B as a q-parallel ring A_q of A, where q depends on curvatures $k(x, y)$ and $h(x, y)$, where $(x, y) \in G \supset A_q$. His bound for the period ω, which is an improvement of the bound given by Diliberto [1], again depends on $||X||$, curvatures $k(x, y)$ and $h(x, y)$ and area of the q-parallel ring A_q. In this paper we shall give a method to obtain upper and, generally, even nontrivial lower bounds for the lengths and periods of periodic solutions of (1) lying in a bounded doubly connected region A. Often the results obtained will be better than those obtained by the method of Lau [2]. The upper and lower bounds of ω obtained by our method are solely functions of $||X||$ on A, of area A and of $k(x, y)$ on A_e, where A_e denote the simply connected region which is bounded by the exterior boundary ∂A_e of A.

Let A_Γ denote the region bounded by the closed trajectory Γ. Suppose that the functions $P(x, y)$ and $Q(x, y)$ belong to class C^1 for all $(x, y) \in \tilde{A} \supset A_\Gamma$ and that the number of singular points of the system (1) is finite in $\tilde{A}$. We state the following theorem:

THEOREM.

$$\left|\iint_{A_\Gamma} k(x, y)\, dx\, dy\right| = L,$$

where L denotes the length of Γ.

Proof. First we remark that the curvature

$$k(x, y) \equiv \frac{P^2Q_x + PQ(Q_y - P_x) - Q^2P_y}{(P^2 + Q^2)^{3/2}}$$

may be written as

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$$k(x, y) \equiv \frac{\partial}{\partial x}[Q(Q^2 + P^2)^{-1/2}] + \frac{\partial}{\partial y}[-P(Q^2 + P^2)^{-1/2}] \equiv \operatorname{div} \mathbf{a}, \tag{2}$$

where

$$\mathbf{a} \equiv \{Q(Q^2 + P^2)^{-1/2}, -P(Q^2 + P^2)^{-1/2}\}. \tag{3}$$

Let S_i, $i = 1, 2, \cdots, n$, denote the singular points of (1) in A_Γ and $K_\epsilon(S_i)$ the circle with center S_i and radius ϵ. Let ϵ be sufficiently small that the relation $\bigcup_{i=1}^n K_\epsilon(S_i) \cap \partial A_\Gamma = \varnothing$ is valid ($\varnothing$: empty set, ∂A_Γ : boundary of A_Γ). Thus in the region $A_\Gamma \backslash \bigcup_{i=1}^n K_\epsilon(S_i)$ all suppositions of the Gaussian theorem are fulfilled. Applying this theorem and (2) we obtain

$$\begin{aligned} \iint_{A_\Gamma \backslash \bigcup_{i=1}^n K_\epsilon(S_i)} k(x, y)\, dx\, dy &= \iint_{A_\Gamma \backslash \bigcup_{i=1}^n K_\epsilon(S_i)} \operatorname{div} \mathbf{a}\, dx\, dy \\ &= \oint_\Gamma \mathbf{a} \cdot \mathbf{n}_\Gamma\, ds + \sum_{i=1}^n \oint_{K_\epsilon(S_i)} \mathbf{a} \cdot \mathbf{n}_i\, ds, \end{aligned} \tag{4}$$

where $\mathbf{n}_\Gamma$ and $\mathbf{n}_i$ denote the unit vectors in the direction of the outward normals of A_Γ and $K_\epsilon(A_i)$ respectively.

According to the definition of $\mathbf{a}$, the following relation is valid:

$$\mathbf{a} \cdot \mathbf{n}_i = ||\mathbf{a}||\, ||\mathbf{n}_i|| \cos(\mathbf{a}, \mathbf{n}_i) = \cos(\mathbf{a}, \mathbf{n}_i).$$

Thus, because of $|\cos(\mathbf{a}, \mathbf{n}_i)| \leq 1$, we have

$$\begin{aligned} \lim_{\epsilon \to 0} \left| \sum_{i=1}^n \oint_{K_\epsilon(S_i)} \mathbf{a} \cdot \mathbf{n}_i\, ds \right| &= \lim_{\epsilon \to 0} \left| \sum_{i=1}^n \oint_{K_\epsilon(S_i)} \cos(\mathbf{a}, \mathbf{n}_i)\, ds \right| \\ &\leq \lim_{\epsilon \to 0} \sum_{i=1}^n \left| \oint_{K_\epsilon(S_i)} ds \right| = \lim_{\epsilon \to 0} 2\pi n \cdot \epsilon = 0. \end{aligned}$$

Therefore we obtain from (4)

$$\iint_{A_\Gamma} k(x, y)\, dx\, dy = \oint_\Gamma \mathbf{a} \cdot \mathbf{n}_\Gamma\, ds. \tag{5}$$

Because Γ is a closed trajectory of the system (1) and $\mathbf{n}_\Gamma$ is the unit vector in the direction of the outward normal of A_Γ, we have $\mathbf{a} \cdot \mathbf{n} = \pm 1$. Consequently, we obtain from (5)

$$\left| \iint_{A_\Gamma} k(x, y)\, dx\, dy \right| = \left| \oint_\Gamma ds \right| = L.$$

This theorem provides us the possibility of estimating the length L and the period ω of Γ.

Let A be a doubly connected region containing the closed trajectory Γ. We suppose that the interior boundary ∂A_i and the exterior boundary ∂A_e of A, bounding the simply connected regions A_i and A_e, are composed of a finite number of smooth arcs. In addition we assume that A is free of singular points of the system (1) and $A_e \subset \tilde{A}$.

Using the notations

$$k = \max_{(x,y) \in A} |k(x, y)|, \qquad |A| = \text{area } A$$

we estimate

$$L = \left|\iint_{A_\Gamma} k(x, y)\, dx\, dy\right| \le \left|\iint_{A_i} k(x, y)\, dx\, dy + \iint_{A_\Gamma \setminus A_i} k(x, y)\, dx\, dy\right|$$
$$< \left|\iint_{A_i} k(x, y)\, dx\, dy\right| + k\, |A|,$$

and

$$L \le \left|\iint_{A_e} k(x, y)\, dx\, dy - \iint_{A_e \setminus A_\Gamma} k(x, y)\, dx\, dy\right| < \left|\iint_{A_e} k(x, y)\, dx\, dy\right| + k\, |A|.$$

Analogously, we can obtain a lower bound for L. Consequently, the following corollary is valid:

COROLLARY 1.

$$L < L_2 = \min\left\{\left|\iint_{A_i} k(x, y)\, dx\, dy\right|, \left|\iint_{A_e} k(x, y)\, dx\, dy\right|\right\} + k\, |A|,$$

$$L > \begin{cases} L_1 = \max\left\{\left|\iint_{A_i} k(x, y)\, dx\, dy\right|, \left|\iint_{A_e} k(x, y)\, dx\, dy\right|\right\} - k\, |A|, \\ \qquad\qquad \textit{if this expression is greater than } |\partial A_i|, \\ |\partial A_i| \quad \textit{otherwise.} \end{cases}$$

If $k(x, y)$ possesses the same sign in $A_e \setminus \bigcup_{i=1}^{n} S_i$, we have

$$\bar{L}_1 = \min\left\{\left|\iint_{A_i} k(x, y)\, dx\, dy\right|, \left|\iint_{A_e} k(x, y)\, dx\, dy\right|\right\} < L$$
$$< \max\left\{\left|\iint_{A_i} k(x, y)\, dx\, dy\right|, \left|\iint_{A_e} k(x, y)\, dx\, dy\right|\right\} = \bar{L}_2$$

Let be s the arc length on Γ. By means of the notations

$$m_i = \min_{(x,y)\in A} \{[P(x, y)]^2 + [Q(x, y)]^2\}^{1/2} = \min_{(x,y)\in A} ||X(x, y)||,$$
$$m_a = \max_{(x,y)\in A} ||X(x, y)||$$

we obtain from relation

$$|ds/dt| = ||X(x, y)||_{|\Gamma}$$

COROLLARY 2. *For the period ω of the closed trajectory, the following estimations are valid*:

$$\omega < \frac{1}{m_i}\left[\min\left\{\left|\iint_{A_i} k(x, y)\, dx\, dy\right|, \left|\iint_{A_e} k(x, y)\, dx\, dy\right|\right\} + k\, |A|\right],$$

$$\omega > \begin{cases} \dfrac{1}{m_a}\left[\max\left\{\left|\iint_{A_i} k(x, y)\, dx\, dy\right|, \left|\iint_{A_e} k(x, y)\, dx\, dy\right|\right\} - k\, |A|\right], \\ \qquad\qquad \textit{if this expression is greater than } |\partial A_i| m_a^{-1}, \\ |\partial A_i| m_a^{-1} \quad \textit{otherwise.} \end{cases}$$

If $k(x, y)$ posseses the same sign in $A_\epsilon \backslash \bigcup_{i=1}^n S_i$, we have

$$\frac{1}{m_a} \cdot \bar{L}_1 < \omega < \frac{1}{m_i} \cdot \bar{L}_2 .$$

For illustrating our results we first consider the very simple system

$$dx/dt = -y, \qquad dy/dt = x.$$

We wish to obtain estimations of the length L and of the period ω of the closed trajectory Γ passing through the point $x = 0$, $y = 1$. It is known that the relations $L = 2\pi$ and $\omega = 2\pi$ are valid.

Let us suppose that the region A containing the closed trajectory Γ is represented by the region which is bounded by the circles $C_1 : x^2 + y^2 = (0.9)^2$ and $C_2 : x^2 + y^2 = (1.1)^2$. Since $k(x, y) = (x^2 + y^2)^{-1/2}$ possesses the same sign in $A_\epsilon \backslash \{0\}$, we obtain from Corollary 1 the estimation $2\pi \cdot (0.9)/(1.1) \leq \omega \leq 2\pi \cdot (1.1)/(0.9)$. The application of the method of Lau [2] yields $L \leq 2\pi \cdot (1 + (0.1)/p) = \tilde{L}_2$ with $p < 0.9$. A comparison of $\bar{L}_2$ and $\tilde{L}_2$ demonstrates $\bar{L}_2 < \tilde{L}_2$. As a second example we consider the system

$$dx/dt = -y + x[(x^2 + y^2)^{1/2} - 1], \qquad dy/dt = x + y[(x^2 + y^2)^{1/2} - 1].$$

This system posseses only one nontrivial periodic solution. The corresponding closed trajectory Γ is represented by the circle $x^2 + y^2 = 1$ with $L = 2\pi$ and $\omega = 2\pi$.

Let us again suppose that the region A containing Γ is formed by the region bounded by C_1 and C_2. Since the function $a(r)$,

$$a(r) = \iint_{A_r} k(x, y)\, dx\, dy = \iint_{A_r} \frac{2 - (x^2 + y^2)}{(1 + [(x^2 + y^2]^{1/2} - 1)^2)^{3/2}}\, dx\, dy = \frac{2\pi r}{[1 + (r - 1)^2]^{1/2}}$$

where A_r is the simply connected region bounded by the circle $x^2 + y^2 = r^2$, is monotone, increasing for $0.9 \leq r \leq 1.1$, we have the estimations

$$2\pi \cdot 0.9 \leq (1.01)^{1/2} \cdot L \leq 2\pi \cdot 1.1 = \bar{L}_2 \cdot (1.01)^{1/2},$$

$$2\pi \cdot \frac{0.9}{1.1} \leq 1.01\omega \leq 2\pi \cdot \frac{1.1}{0.9} .$$

If we use the method of Lau we obtain $L \leq 2\pi(1 + 0.1 \cdot p^{-1}) = \tilde{L}_2$, $p < 0.9$. A comparison again yields $\bar{L}_2 < \tilde{L}_2$.

References

[1] S. P. Diliberto, "Bounds for periods of periodic solutions" in *Contributions to the theory of nonlinear oscillations*, Vol. III, Annals of Math. Studies, no. 36, Princeton Univ. Press, Princeton, N. J., 1955, pp. 269–275

[2] P. J. Lau, *Bounds for the lengths and periods of closed orbits of two-dimensional autonomous systems of differential equations*, J. Differential Eqs. **3**, 330–342 (1967)