CHARACTERIZATION OF SINGLE-BLOW TEMPERATURE RESPONSES BY FIRST MOMENTS*

BY

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Abstract. Transient temperature response functions, given as analytic solutions of the single-blow problem in transient heat transfer analysis, are characterized by the first moment of the difference between downstream and upstream fluid temperatures. Both the single-blow problem and the corresponding inverse problem are formulated in terms of Volterra integral equations. Monotonicity and boundedness properties of the response functions are derived. It is shown that the first moment of the temperature difference is well suited for an indirect solution of the curve matching problem.

1. Introduction. The transient heat-transfer testing of porous media is frequently based on the analysis of the so-called single-blow problem which has been discussed by several authors [1]–[9]. In the single-blow experiment one subjects the test core, after a stationary temperature distribution has been attained in it, to some change of the upstream temperature of the transfusing fluid. Both the fluid temperature at the test-section inlet, the *temperature stimulus*, and the fluid temperature at the test section exit, the *transient response*, are recorded, and the problem is to infer, on the basis of the mathematical model for the single-blow problem, the number of heat transfer units from these recorded data.

The basic mathematical model, due to Hausen [1] and Schumann [2], which describes the heat transfer to and from a fluid transfusing through a porous medium, is a hyperbolic system of two linear partial differential equations of first order. This system was somewhat modified by this author [7] for the specific purpose of efficient test data reduction; it is restated here (with a liberal attitude toward a more standard nomenclature) as

$$\frac{\partial G}{\partial x} + G(x, t) - S(x, t) = 0, \tag{1}$$

$$\frac{\partial S}{\partial t} + aS(x, t) - aG(x, t) = 0. \tag{2}$$

In Eqs. (1) and (2), G stands for the temperature of the fluid (gas) and S for the temperature of the solid; the independent variables are x, a length-like (dimensionless) variable, and t, a time-like (dimensionless) variable. A dual role is played by the positive constant a: on the one hand it determines the interval $0 \leqq x \leqq a$ over which x is allowed to vary, on the other it denotes the *number of heat transfer units* (usually denoted by N_{tu}, a measure for the "size", or transfer capacity, of the test core) which enters in the solution of Eqs. (1) and (2) as a parameter. The reader looking for background information is referred to [7].

The mathematical model becomes well posed as a *Goursat problem* by the stipulation

*Received May 1, 1967.

of appropriate boundary and initial conditions, namely

$$G(x, t)\,|_{x=0} = g(t) \tag{3}$$

and

$$S(x, t)\,|_{t=0} = s(x). \tag{4}$$

According to Eqs. (3) and (4), the *Goursat data* are given in terms of the temperature stimulus $g(t)$ and the initial solid temperature $s(x)$. For that particular case where the test-core is subjected to cooling, and the ultimate temperature difference is normalized, one can set $s(x) \equiv 1$. Only the latter case will be considered in this paper.

An analytical solution of Eqs. (1)–(4) can be obtained by several standard methods (see, e.g., [8]). Only the known analytical solution for the fluid temperature G will be needed in the following considerations; the solid temperature S can be eliminated. The object of this paper is to show how the known analytical solution G of the single-blow problem can be utilized to solve the so-called *curve matching problem*, that is the problem of determining the unknown parameter a characteristic for a particular test core by matching theoretical and experimental temperature response functions. This problem is, even though the content of physical terms is prominent, primarily a mathematical one.

The dependence of the transient response on the temperature stimulus can be exhibited, very clearly and compactly, in terms of a Volterra integral equation of the second kind. From this integral equation one can easily deduce the dependence of the stimulus on the response; this dependence is the subject of the "backward single-blow problem," or "retrodiction problem." The solution of the latter is required for the analysis of the curve matching problem, and this is the major reason for its inclusion in this paper, but it may, nevertheless, be worthwhile to point out that it has also applications in control theory.

We will show that the first moment of the difference between response and stimulus, $\int_0^\infty t(G - g)\,dt$, is a suitable functional on which a simple indirect curve matching method can be based. In the way of preparation, a few results concerning the properties of, and the relationship between, stimulus and response, such as monotonicity, boundedness, and invariance properties, will be stated and proved. Some of these results are more or less taken for granted by the experimentalist, but their deduction from the mathematical model has not been found in the literature.

2. Integral equation formulations. In this section we will discuss two integral equation formulations of the single-blow problem as described by Eqs. (1)–(4). The second of these, Eq. (11), is believed to be new.

The characteristics of the hyperbolic system of partial differential equations (1) and (2) coincide with the coordinate lines $x = \text{const}$ and $y = \text{const}$; Eqs. (1) and (2) can therefore be integrated as ordinary differential equations. Upon elimination of the less interesting solid temperature $S(x, t)$ from the resulting integral equation system, we obtain the integral equation

$$G(x, t) - a \int_{\xi=0}^{x} \int_{\theta=0}^{t} e^{-(x-\xi)-a(t-\theta)} G(\xi, \theta)\, d\xi\, d\theta = g(t)e^{-x} + (1 - e^{-x})e^{-at}, \tag{5}$$

which is of Volterra type, with respect to both independent variables, x and t. Equation (5) has a unique solution which can be found by several standard methods. This solution will be referred to as transient temperature response function, or simply transient response. It is clear that the transient response $G(x, t)$ depends on the temperature stimulus $g(t)$. We will be concerned with the nature of this dependence.

The experimentalist usually restricts his attention to the transient response at $x = a$; that is, he measures the response only at the exit cross section of the test core. This suggests the question of whether the mathematical model as expressed by Eq. (5) can likewise be simplified by considering the case $x = a$ only. To show that this can indeed be done, we restate Eq. (5) in terms of its Laplace transform representation with respect to t (using the notation $\bar{g}(s) = \mathfrak{L}\{g(t); s\} = \int_0^\infty e^{-st}g(t)\,dt$, etc.):

$$\bar{G}(x, s) - a\int_0^x e^{-(x-\xi)}\bar{G}(\xi, s)\,d\xi/(s + a) = \bar{g}(s)e^{-x} + (1 - e^{-x})/(s + a). \tag{6}$$

As can easily be verified, the integral equation (6) has a unique solution, namely

$$\bar{G}(x, s) = \bar{g}(s)\exp[-sx/(s + a)] + 1/s\{1 - \exp[-sx/(s + a)]\}. \tag{7}$$

Upon setting $x = a$ in Eq. (7) and rearranging some terms we obtain the Laplace transform representation of the dependence of response on stimulus.

$$\bar{G}(a, s) = 1/s + [\bar{g}(s) - 1/s]\exp(-a + a^2/(s + a)). \tag{8}$$

There are two uses of Eq. (8). Before discussing these, we find it convenient to introduce a family of entire functions $\Xi_k(\cdot)$ which are defined by $(k = 0, 1, 2, \cdots)$

$$\Xi_k(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!\,(n + k)!} = \begin{cases} I_k(2(x)^{1/2})/(x^{1/2})^k & (x \geqq 0), \\ J_k(2\,|x|^{1/2})/(|x|^{1/2})^k & (x \leqq 0), \end{cases} \tag{9}$$

and seen to be related to the Bessel and modified Bessel functions of the first kind and order k.

To demonstrate the first use of Eq. (8), we re-express it as

$$\exp\left(-\frac{a^2}{s + a}\right)\bar{G}(a, s) = \bar{g}(s)\,e^{-a} + \frac{1}{s}\left[\exp\left(-\frac{a^2}{s + a}\right) - e^{-a}\right]. \tag{10}$$

Using the fact that $\mathfrak{L}\{\Xi_1(-a^2t)e^{-at}; s\} = \exp[-a^2/(s + a)] - 1$, we see that Eq. (10) is the Laplace transform representation of the integral equation

$$G(a, t) - a^2\int_0^t \Xi_1[-a^2(t - \theta)]e^{-a(t-\theta)}G(a, \theta)\,d\theta$$
$$= e^{-a}[g(t) - 1] + a^2\int_0^t \Xi_1(-a^2\theta)e^{-a\theta}\,d\theta, \tag{11}$$

which describes the single-blow problem restricted by $x = a$. The Volterra integral equation (11) is simpler than Eq. (5) because it involves only one independent variable. Moreover, the significance of Eq. (11) goes beyond a purely formal one: it can be put to practical use when the numerical evaluation of the transient response is required.

The second, more immediate and more obvious use of Eq. (8) is, of course, the calculation of the transient response. Laplace transform inversion leads to the analytical solution of Eq. (11)

$$G(a, t) = 1 - [1 - g(t)]e^{-a} - a^2e^{-a}\int_0^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}[1 - g(\theta)]\,d\theta \tag{12}$$

which is known from earlier publications [7]–[9], but restated here because it will be needed in the following sections.

3. Curve matching by functionals. Curve matching is essentially a problem of parameter identification; what is required is the computation of that parameter for which "matching" is achieved. Along with *theoretical* response functions, we will consider *experimental* response functions which we will denote by $G_{\exp}(t)$. We assume that both $G_{\exp}(t)$ and $tG_{\exp}(t)$ have finite L^1-norms, i.e. $\int_0^\infty |G_{\exp}(t)|\,dt < \infty$ and $\int_0^\infty |tG_{\exp}(t)|\,dt < \infty$.

Direct curve matching offers conceptually a simple and reliable approach: by defining a suitable *distance functional*, $d[G_{\exp}(t), G(a, t)]$, the curve matching problem can be reduced to finding that element $G(a, t)$ in the metric space of response functions for which the distance assumes a minimum. (An example of a useful distance functional is $d(a) = \{\int_0^\infty [G_{\exp}(t) - G(a, t)]\,dt\}^{1/2}$.) Direct curve matching is, however, not always desirable because of the considerable computational effort necessitated. For this reason, one looks for an alternate approach.

In *indirect curve matching* the problem is reduced to that of assigning to each theoretical response $G(a, t)$ a real number $\varphi(a)$ by defining a suitable real single-valued continuous functional Φ: $\{G(a, t)\} \to \{\varphi(a)\}$ from the one-parameter family of theoretical response functions into the real line. This functional is so chosen that its values, $\Phi(G) = \varphi(a)$, determine a real single-valued strictly monotone continuous function φ: $\{a\} \to \{\varphi(a)\}$ from the domain of the parametric variable a into the real line. The domain of φ is the real half-line $0 \leqq a < \infty$.

The domain of Φ can be extended to include the experimental response as well. Thus Φ is extended to include the mapping $G_{\exp}(t) \to \varphi_{\exp}$; the value of Φ corresponding to $G_{\exp}(t)$ is then given by $\Phi(G_{\exp}) = \varphi_{\exp}$. Knowing $\varphi(\cdot)$, one can compute the parameter a by first evaluating $\varphi_{\exp}$ and then evaluating the function inverse to $\varphi(\cdot)$, namely $\varphi^{-1}(\cdot)$, at the fixed real argument $\varphi_{\exp}$, according to the formula

$$a = \varphi^{-1}(\varphi_{\exp}). \tag{13}$$

The determination of the parameter a reduces to a problem of interpolation if the inverse function $\varphi^{-1}(\cdot)$ is not given explicitly. No general rule seems to be known for designing suitable functionals for indirect curve matching.

4. Monotonicity and boundedness properties. Before we proceed to introduce a suitable functional, it may be instructive to discuss a few properties of the solution of Eq. (11). We will need

LEMMA 1. *Let θ, t, a all be nonnegative real numbers, suppose $\Xi_1(\cdot)$ is defined as in Eq.* (9), *and let*

$$G_h(a, t) = e^{-a}\left[1 + a^2 \int_0^t \Xi_1(a^2\theta)e^{-a\theta}\,d\theta\right];$$

then the inequality

$$0 \leqq G_h(a.\ t) \leqq 1 \tag{14}$$

holds.

Proof. Nonnegativity of the integrand, $\Xi_1(a^2\theta)e^{-a\theta} \geqq 0$, implies $G_h(a, t) \geqq 0$. For all nonnegative integers n one has $0 \leqq (n!)^{-1}\int_0^t \theta^n e^{-a\theta}\,d\theta \leqq a^{-(n+1)}$. Recalling the definition of $\Xi_1(\cdot)$, one finds

$$G_h(a, t) = e^{-a}\left[1 + a^2 \sum_{n=0}^{\infty} \frac{a^{2n}}{(n+1)!\,n!} \int_0^{\infty} \theta^n e^{-a\theta}\,d\theta\right]$$
$$\leqq e^{-a}\left[1 + \sum_{n=0}^{\infty} \frac{a^{n+1}}{(n+1)!}\right] = e^{-a}e^{a} = 1.$$

Remark. $G_h(a, t)$ represents the temperature response in the solid test core due to a unit step-change (sudden heating) of magnitude one. The temperature response $G_h(a, t)$ is thus bounded by the value of the temperature stimulus before the step ($g(0-) = 0$), and after the step ($g(0+) = 1$).

The importance of arranging a single-blow test such that the temperature stimulus $g(t)$ is monotone is brought out by

Theorem 1. *Let $g(t)$ for all $t \geqq 0$ be bounded by $0 \leqq g(t) \leqq 1$ and monotone decreasing; then the response $G(a, t)$ as given by Eq.* (12) *is*

(i) *also montone decreasing, and*
(ii) *exceeds or equals $g(t)$.*

Proof. To prove (i) it suffices to show that $G(a, t_1) \geqq G(a, t_2)$ for all $a \geqq 0$ and $t_2 \geqq t_1 \geqq 0$. To this end, note first that

$$\int_0^t \Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}[g(\theta) - 1]\,d\theta = \int_0^t \Xi_1(a^2\theta)e^{-a\theta}[g(t-\theta) - 1]\,d\theta.$$

Consider next

$$G(a, t_2) - G(a, t_1) = e^{-a}[g(t_2) - g(t_1)] + a^2e^{-a}\left\{\int_0^{t_1} \Xi_1(a^2\theta)e^{-a\theta}[g(t_2-\theta) - g(t_1-\theta)]\,d\theta\right.$$
$$\left. + \int_0^{t_2} \Xi_1(a^2\theta)e^{-a\theta}[g(t_2-\theta) - 1]\,d\theta\right\}.$$

We have $\Xi_1(a^2\theta)e^{-a\theta} \geqq 0$ for all $a \geqq 0$, $\theta \geqq 0$ and $g(t_2 - \theta) - 1 \leqq 0$; monotonicity of $g(t)$ implies $g(t_1) \geqq g(t_2)$ and hence $g(t_2 - \theta) - g(t_1 - \theta) \leqq 0$ for $0 \leqq \theta \leqq t_1$. Thus one obtains $G(a, t_2) - G(a, t_1) \leqq 0$.

To prove (ii) it suffices to show that $G(a, t) - g(t)$ is nonnegative. The last difference can be written as

$$G(a, t) - g(t) = [1 - g(t)](1 - e^{-a}) - a^2e^{-a}\int_0^t \Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}[1 - g(\theta)]\,d\theta.$$

From $1 - g(t) \geqq 1 - g(\theta)$ follows

$$[1 - g(t)]\int_0^t \Xi_1(a^2\theta)e^{-a\theta}\,d\theta \geqq \int_0^t \Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}[1 - g(\theta)]\,d\theta$$

and consequently

$$G(a, t) - g(t) \geqq [1 - g(t)]\left\{1 - e^{-a}\left[1 + a^2\int_0^t \Xi_1(a^2\theta)e^{-a\theta}\,d\theta\right]\right\};$$

but the inequality (14) of Lemma 1 implies $0 \leqq 1 - G_h(a, t) \leqq 1$ so that $G(a, t) - g(t) \geqq 0$.

The fact that upper and lower bounds on a temperature stimulus give rise to transient response functions with corresponding boundedness properties is manifested in

THEOREM 2. *Let $g_0(t)$, $g_1(t)$, $g_2(t)$ and $g_3(t)$ satisfy the conditions of Theorem* 1 *and suppose that $g_0(t) \equiv 0$ and $g_3(t) \equiv 1$ for all $t > 0$, and that the inequality $0 \leqq g_1 \leqq g_2 \leqq 1$ is valid for all $t \geqq 0$. Then the corresponding responses $G_0(a, t)$, $G_1(a, t)$, $G_2(a, t)$ and $G_3(a, t)$, all determined by Eq.* (12), *satisfy the inequality*

$$1 - G_h(a, t) = G_0(a, t) \leqq G_1(a, t) \leqq G_2(a, t) \leqq G_3(a, t) = 1$$

for all $a \geqq 0$, $t \geqq 0$.

Proof. First, consider only G_1 and G_2; to prove $G_1 \leqq G_2$ it suffices to show that $G_2 - G_1$ is nonnegative. The assumption that $g_2 - g_1 \geqq 0$ implies

$$G_2(a, t) - G_1(a, t)$$
$$= e^{-a}\left\{[g_2(t) - g_1(t)] + a^2 \int_0^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}[g_2(\theta) - g_1(\theta)]\, d\theta\right\} \geqq 0. \tag{15}$$

Next, determine the responses due to g_0 and g_3. To g_0 corresponds, according to Eq. (12), the response $G_0(a, t) = 1 - G_h(a(a, t)$ for all $t \geqq 0$; as the indices are arbitrary, Eq. (15) implies $G_1 \geqq G_0$. Finally one obtains, again from Eq. (12), the response due to g_3 as $G_3(a, t) = 1$ for all $t \geqq 0$, and Eq. (15) implies $G_3 \geqq G_2$.

REMARK. The response $G_0(a, t)$ due to the step-function $g_0(t) = 1$, $t = 0$, $g_0(t) = 0$, $t > 0$, when evaluated at $t = 0$ is given by $G_0(a, 0) = 1 - [1 - g_0(t)]e^{-a} = 1$ and has thus at $t = 0$ a jump-discontinuity of magnitude e^{-a}.

5. Evaluation of moments. From the moment problem it is known that a continuous function can be characterized by the totality of all of its moments, provided certain conditions are satisfied. (See, for example, [10].) This suggests the question of whether transient response functions can be characterized by the moments of response and stimulus. Attempting not to go into detail, or to become too technical, we point out here that the infinite sequence $M = \{M_h(f)\}_{n=0}^{\infty}$ of moments of order n, $M_n(f) = \int_0^\infty x^n f(x)\, dx$, determines uniquely the function f whose moments are known. This becomes evident after forming the expression $\sum_{n=0}^{\infty} (-s)^n M_n(f) = \int_0^\infty e^{-sx} f(x)\, dx$ which is recognized as the Laplace transform of f. It is well known that the latter can be uniquely inverted [11].

Since the transient response functions $G(a, t)$ constitute a one-parameter family continuous in the parameter a, it follows that not all elements of the infinite moment sequence are required for the characterization. Only low order moments seem to be attractive as suitable functionals for curve matching. To be practical, we will consider only zero and first order moments.

The transient response due to a unit step function is given by

$$\begin{aligned} G_0(a, t) &= 1 - G_h(a, t) \\ &= 1 - e^{-a}\left[1 + a^2 \int_0^\infty \Xi_1(a^2\theta)e^{-a\theta}\, d\theta\right]. \end{aligned} \tag{16}$$

For this particular response, two simple formulas for the evaluation of the zero order and first order moments are stated in the following lemma.

LEMMA 2. *Let the transient response G_0 be determined by Eq.* (16) *and suppose $0 < a < \infty$. Then the first two moments of G_0 are finite and given by*

(i) $$\int_0^\infty G_0(a, t)\, dt = 1,$$

(ii) $$\int_0^\infty tG_0(a, t)\, dt = 1/2 + 1/a.$$

Proof. It is of considerable advantage to employ Laplace transforms. Note that both G_0 and tG_0 are nonnegative. The Laplace transform of G_0, $\mathfrak{L}\{G_0(a, t); s\} = (1/s)\{1 - e^{-a} \exp[a^2/(s + a)]\}$, converges for any $s > 0$. The limit of the last expression as $s \to 0+$ exists and can be determined according to L'Hospital's rule:

$$\begin{aligned}\lim_{s\to 0+} \frac{1}{s}\{1 - e^{-a}\exp[a^2/(s + a)]\} &= \lim_{s\to 0+} \frac{d}{ds}\{1 - e^{-a}\exp[a^2/(s + a)]\} \\ &= [a^2e^{-a}/(s + a)^2]\exp[a^2/(s + a)]\,|_{s=0} = 1.\end{aligned} \tag{17}$$

The Tauberian theorem for Laplace transforms [12, p. 195] applies and yields $\lim_{s\to 0+} \mathfrak{L}\{G_0; s\} = \lim_{t\to\infty}\int_0^t G_0(a, \theta)\, d\theta = \int_0^\infty G_0(a, t)\, dt$; thus (i) is proved. To prove (ii), note that

$$\begin{aligned}\mathfrak{L}\{tG_0(a, t); s\} &= -\frac{d}{ds}\mathfrak{L}\{G_0; s\} \\ &= \frac{1}{s^2}\{1 - e^{-a}\exp[a^2/(s + a)] - [sa^2e^{-a}/(s + a)^2]\exp[a^2/(s + a)]\}\end{aligned}$$

and use again L'Hospital's rule and the Tauberian theorem. There results

$$\begin{aligned}&\lim_{s\to 0+}\left(-\frac{d}{ds}\mathfrak{L}\{G_0; s\}\right) \\ &\quad = \lim_{s\to 0+}\left(\frac{1}{2s}\frac{d}{ds}\{1 - e^{-a}\exp[a^2/(s + a)] - [sa^2e^{-a}/(s + a)^2]\exp[a^2/(s + a)]\}\right) \\ &\quad = \lim_{s\to 0+}\{[a^2e^{-a}(a^2 + 2a + 2s)/2(s + a)^4]\exp[a^2/(s + a)]\} = 1/2 + 1/a.\end{aligned}$$

Of greater interest than the case of a step function stimulus is that of an "arbitrary" temperature stimulus $g(t)$, for which the result is stated in

THEOREM 3. *Let both the temperature stimulus $g(t)$ as well as the function $tg(t)$ have finite $L^1(0, \infty)$-norms. Assume that the transient response $G(a, t)$ due to $g(t)$ is determined by Eq.* (12) *and suppose $0 < a < \infty$. Then the first two moments of the temperature difference $G(a, t) - g(t)$ are finite and given by*

(i) $$\int_0^\infty [G(a, t) - g(t)]\, dt = 1,$$

(ii) $$\int_0^\infty t[G(a, t) - g(t)]\, dt = 1/2 + 1/a + \int_0^\infty g(t)\, dt.$$

Proof. Represent the transient response by

$$G(a, t) = G_0(a, t) + e^{-a}\left\{g(t) + a^2\int_0^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}g(\theta)\, d\theta\right\}. \tag{18}$$

We will need the auxiliary result that the kernel of the integral in Eq. (18), $\Xi_1(a^2t)e^{-at}$, is an $L^1(0, \infty)$-function. To see this, note that the kernel is nonnegative, that its Laplace transform

$$\mathcal{L}\{a^2\Xi_1(a^2t)e^{-at}; s\} = \sum_{n=0}^{\infty} a^{2(n+1)}/(n+1)!\,(s+a)^{n+1} = \exp\,[a^2/(s+a)] - 1$$

is convergent for any $s > 0$ and that the limit of the latter, $\lim_{s\to0+}\{a^2\Xi_1(a^2t)e^{-at}; s\} = e^a - 1$, exists and is finite. The Tauberian theorem for Laplace transforms [12, p. 195] holds under these conditions and one obtains

$$0 < \int_0^\infty a^2\Xi_1(a^2t)e^{-at}\,dt = \int_0^\infty |a^2\Xi_1(a^2t)e^{-at}|\,dt = e^a - 1 < \infty,$$

so that $\Xi_1(a^2t)e^{-at} \in L^1(0, \infty)$.

To prove part (i) of Theorem 3, note that the convolution integral

$$[a^2\Xi_1(a^2t)e^{-at}] * g(t) = \int_0^t a^2\Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}g(\theta)\,d\theta$$

has a finite L^1-norm since both $\Xi_1(a^2t)e^{-at}$ and $g(t)$ are L^1-functions. Owing to formula (i) of Lemma 2, and because it is nonnegative, G_0 has a finite L^1-norm. Hence G is also an L^1-function and, by Hölder's inequality, we have

$$\int_0^\infty e^{-st}G(a, t)\,dt \leqq \int_0^\infty |G(a, t)|\,dt < \infty \qquad \text{for all} \quad s \geqq 0.$$

The last integral inequality assures that

$$\lim_{s\to0+} \mathcal{L}\{G(a, t); s\} = \int_0^\infty G(a, t)\,dt. \tag{19}$$

Consider now the Laplace transform of G,

$$\mathcal{L}\{G; s\} = \mathcal{L}\{G_0\,; s\} + e^{-a}\exp\,[a^2/(s+a)]\mathcal{L}\{g; s\}. \tag{20}$$

According to the Abelian theorem for Laplace transforms one has

$$\lim_{s\to0+} \mathcal{L}\{g(t); s\} = \int_0^\infty g(t)\,dt. \tag{21}$$

Take the limit of Eq. (20) as $s \to 0+$ and note Eqs. (17) and (21). Thus one obtains

$$\lim_{s\to0+} \mathcal{L}\{G; s\} = 1 + \int_0^\infty g(t)\,dt, \tag{22}$$

and by combining Eqs. (19) and (22) part (i) is proved.

To prove part (ii) of Theorem 3 it suffices to show that tG is an L^1-function. To do so, we have to show first that the nonnegative function $t\Xi_1(a^2t)e^{-at}$ has a finite L^1-norm. This is made evident by studying the Laplace transform $\mathcal{L}\{ta^2\Xi_1(a^2t)e^{-at}; s\} = [a^2/(s+a)]\exp\,[a^2/(s+a)]$, which converges for any $s > 0$ and has the finite limit

$$\lim_{s\to0+}\mathcal{L}\{ta^2\Xi_1(a^2t)e^{-at}; s\} = e^a.$$

The Tauberian theorem for Laplace transforms holds and one obtains $\int_0^\infty ta^2\Xi_1(a^2t)e^{-at}\,dt = e^a < \infty$ so that $t\Xi_1(a^2t)e^{-at} \in L^1(0, \infty)$. Next, we note that the function

$$t[\Xi_1(a^2t)e^{-at} * g(t)] = t\int_0^t \Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}g(\theta)\,d\theta$$

$$= \int_0^\infty (t-\theta)\Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}g(\theta)\,d\theta + \int_0^\infty \Xi_1[a^2(t-\theta)]e^{-a(t-\theta)}\theta g(\theta)\,d\theta$$

$$= [t\Xi_1(a^2t)e^{-at}] * g(t) + [\Xi_1(a^2t)e^{-at}] * [tg(t)]$$

has a finite L^1-norm, as $\Xi_1(a^2t)e^{-at}$, $t\Xi_1(a^2t)e^{-at}$, g, and tg are all L^1-functions (as are all sums and convolution integrals of the latter). From part (ii) of Lemma 2 and from the fact that tG_0 is nonnegative follows $tG_0 \in L^1(0, \infty)$. Therefore, tG has finite L^1-norm.

The rest of the proof is straightforward. Application of Hölder's inequality leads, for all $s \geqq 0$, to the inequality

$$\int_0^\infty e^{-st}tG(a, t)\,dt \leqq \int_0^\infty |tG(a, t)|\,dt < \infty,$$

which assures that

$$\lim_{s\to 0+} \mathfrak{L}\{tG(a, t); s\} = \lim_{s\to 0+}\left(-\frac{d}{ds}\mathfrak{L}\{G; s\}\right) = \int_0^\infty tG(a, t)\,dt. \tag{23}$$

Next, differentiate Eq. (20) with respect to s. There follows

$$\frac{d}{ds}\mathfrak{L}\{G; s\} = \frac{d}{ds}\mathfrak{L}\{G_0\,; s\} - [a^2e^{-a}/(s+a)^2]\exp[a^2/(s+a)]\mathfrak{L}\{g; s\}$$

$$+ e^{-a}\exp[a^2/(s+a)]\frac{d}{ds}\mathfrak{L}\{g; s\}. \tag{24}$$

The Abelian theorem for Laplace transforms inplies $\lim_{s\to 0+}(-(d/ds)\mathfrak{L}\{G; s\}) = \int_0^\infty tg(t)\,dt$. On changing the sign, of Eq. (24), taking the limit as $s \to 0+$, and using part (ii) of Lemma 2, there follows

$$\lim_{s\to 0+}\left(-\frac{d}{ds}\mathfrak{L}\{G; s\}\right) = \frac{1}{2} + \frac{1}{a} + \int_0^\infty g(t)\,dt + \int_0^\infty tg(t)\,dt, \tag{25}$$

and by combining Eqs. (23) and (25) part (ii) of Theorem 3 is proved.

6. Choice of functional for curve matching. The result of Theorem 3 is significant in several respects. First of all, it shows that the zero-order moment of $G(a, t)$ namely $\int_0^\infty G(a, t)\,dt$, cannot be used to characterize response functions, because it is independent of the parameter a. Secondly, we recognize that $\int_0^\infty t[G(a, t) - g(t)]\,dt$ is a very natural choice for a curve matching functional. It may come as a surprise that the dependence of this functional on the parameter a is so simple; it is also interesting to note that this functional does not depend on the first moment of the stimulus. Finally, $\int_0^\infty t(G - g)\,dt$ has a very simple geometrical interpretation: it is the centroid coordinate of the area bounded by stimulus and response.

Among the conditions of Theorem 3, monotonicity of $g(t)$ was not required. It is, nevertheless, desirable to assume that $g(t)$ be monotone decreasing. For monotone decreasing $g(t)$ one obtains, after defining $I(g) = \int_0^\infty g\,dt$ and $M_1(g) = \int_0^\infty t(G - g)\,dt$ that $g_2(t) \geqq g_1(t)$ for all $t \geqq 0$ implies $I(g_2) \geqq I(g_1)$ and, furthermore, that $I(g_2) \geqq I(g_1)$ implies and is implied by $M_1(g_2) \geqq M_1(g_1)$. These properties can be useful when error bounds are needed [13].

If $\int_0^\infty t(G - g)\,dt$ is chosen as a curve matching functional, then the solution of the curve matching problem reduces to the computation of the parameter a according to the formula

$$a = \left[\int_0^\infty t[G_{\exp}(t) - g(t)]\,dt - \frac{1}{2} - \int_0^\infty g(t)\,dt\right]^{-1}, \tag{26}$$

which follows from part (ii) of Theorem 3 as the concrete counterpart of Eq. (13). The reader who is interested in the practical aspects of curve matching is referred to [13].

7. Delaying time shifts. Occasionally it is more or less taken for granted that the "shape" of the transient response, due to some fixed temperature stimulus, is independent of the time when the single-blow test is initiated. To deduce this rather important invariance property of the response relative to a (delaying) time shift, the following definition will afford an exact formulation.

Definition 1. *A shift operator $\mathcal{S}_T$ with shift T is a mapping determined by*

$$\mathcal{S}_T g(t) = \begin{cases} 1 & (t \leqq T), \\ g(t - T) & (t > T), \end{cases}$$

where $g(t)$ is any stimulus which is integrable and restricted by $g(0) = 1$ and $0 \leqq g(t) \leqq 1$ for all $t \geqq 0$; the result of this mapping is called delayed temperature stimulus.

We can now state the announced invariance property in the following theorem.

Theorem 4. *Let $\mathcal{S}_T$ be a shift operator such that $0 \leqq T < \infty$. Let $G(a, t; g) = G(a, t)$ be the response due to $g(t)$ as determined by Eq. (12); then*

$$G(a, t; \mathcal{S}_T g) = \mathcal{S}_T G(a, t; g),$$

that is, the response due to a delayed stimulus is equal to the delayed response due to the undelayed stimulus.

Proof. According to Eq. (12) one has

$$G(a, t; \mathcal{S}_T g) = 1 - e^{-a}[\mathcal{S}_T g(t) - 1] + a^2 e^{-a} \int_0^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}[\mathcal{S}_T g(\theta) - 1]\,d\theta.$$

For $t \leqq T$ one obtains $G(a, t; \mathcal{S}_T g) = 1$; for $t > T$, after noting that

$$\int_T^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}[g(t - T) - 1]\,d\theta$$

$$= \int_0^{t-T} \Xi_1[a^2(t - T - \tau)]e^{-a(t-T-\tau)}[g(\tau) - 1]\,d\tau,$$

where $\tau = \theta - T$ is a new variable of integration, one finds

$$G(a, t; \mathcal{S}_T g) = 1 + e^{-a}[g(t - T) - 1]$$

$$+ a^2 e^{-a} \int_0^{t-T} \Xi_1[a^2(t - T - \theta)]e^{-a(t-T-\theta)}[g(\theta) - 1]\,d\theta = G(a, t - T; g).$$

REMARK. For a time delayed stimulus the previously discussed curve matching functional assumes the value

$$\int_0^\infty t[G(a, t; \mathcal{S}_T g) - \mathcal{S}_T g]\, dt = \frac{1}{2} + \frac{1}{a} + \int_0^\infty \mathcal{S}_T g(t)\, dt$$
$$= \frac{1}{2} + \frac{1}{a} + T + \int_0^\infty g(t)\, dt = T + \int_0^\infty t(G - g)\, dt,$$

as one might expect from geometrical considerations.

8. Temperature retrodiction. Whereas it is known that the single-blow problem as formulated by either of the two integral equations (5) or (11) has a unique solution, namely the solution expressed by Eq. (12), there remains nevertheless the question whether a given transient response determines uniquely the temperature stimulus. This is the so-called

Retrodiction problem (or backward single-blow problem). Given
(a) the value of the parameter a and
(b) a transient response function $G(a, t)$ due to some unknown temperature stimulus $g(t)$; find $g(t)$.

To obtain an integral equation representation of the retrodiction problem, we return to Eq. (8), which we rewrite as

$$\exp [a^2/(s + a)]\bar{g}(s) = e^a\left[\bar{G}(a, s) - \frac{1}{s}\right] + \frac{1}{s} \exp [a^2/(s + a)]. \tag{27}$$

Equation (27) is the Laplace transform representation of the Volterra integral equation

$$g(t) + a^2 \int_0^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}g(\theta)\, d\theta = e^a[G(a, t) - 1] + 1 + a^2 \int_0^t \Xi_1(a^2\theta)e^{-a\theta}\, d\theta, \tag{28}$$

which looks rather similar to the integral equation (11) for the single-blow problem.

A solution of Eq. (28) describes that temperature stimulus which produces the prescribed transient response $G(a, t)$. In other words, Eq. (28) provides, when solved, the answer to the question (or the control problem) of how the upstream fluid temperature is to be regulated so that the downstream fluid temperature turn out as prescribed. It may be worthwhile here to point out that the right-hand side of Eq. (28) can be somewhat simplified if one makes use of the expression for the transient response $G_0(a, t)$, due to a unit step function, as given by Eq. (16). Using $G_0(a, t)$ one obtains the integral equation

$$g(t) + a^2 \int_0^t \Xi_1[a^2(t - \theta)]e^{-a(t-\theta)}g(\theta)\, d\theta = e^a[G(a, t) - G_0(a, t)]. \tag{29}$$

To solve Eq. (29), we use again Laplace transforms. From

$$\exp [a^2/(s + a)]\bar{g}(s) = e^a[\bar{G}(a, s) - \bar{G}_0(a, s)]$$

follows

$$\bar{g}(s) = e^a \exp [-a^2/(s + a)][\bar{G}(a, s) - \bar{G}_0(a, s)],$$

the Laplace transform of the solution of the retrodiction problem. The inversion of $\bar{g}(s)$ is straightforward; we obtain

$$g(t) = e^a\left\{[G(a, t) - G_0(a, t)] - a^2 \int_0^t \Xi_1[-a^2(t - \theta)]e^{-a(t-\theta)}[G(a, \theta) - G_0(a, \theta)]\, d\theta\right\}$$

$$= 1 + e^a\left\{G(a, t) - a^2 \int_0^t \Xi_1[-a^2(t - \theta)]e^{-a(t-\theta)}G(a, \theta)\, d\theta\right\}$$

$$- e^a\left[1 - a^2 \int_0^t \Xi_1(-a^2\theta)e^{-a\theta}\, d\theta\right] \qquad (30)$$

as the unique solution of the temperature retrodiction problem. We are thus led to

THEOREM 5. *Suppose $a \geqq 0$ and let both $g(t)$ and $G(a, t)$ be bounded $L^1(0, \infty)$-functions (with respect to t, $t \geqq 0$). Suppose further that $G(a, t)$ is determined by $g(t)$ according to Eq.* (12) *and that $g(t)$ is determined by $G(a, t)$ according to Eq.* (30). *Then the two mappings $\{g\} \to \{G\}$ and $\{G\} \to \{g\}$ are one-to-one.*

REMARK. From Theorem 5 we can deduce the following uniqueness property. Given the (heat-transfer) parameter a, there corresponds to one given transient response but one value of the curve-matching functional. This result justifies post factum the approach chosen in Sec. 3 of this paper, where the curve-matching functional was assumed to depend only on the transient response $G(a, t)$, but not explicitly on the temperature stimulus $g(t)$.

Acknowledgements. The author would like to thank Dr. E. N. Nilson and Dr. A. M. Pfeffer for valuable discussions.

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