DECAY OF THE KINETIC ENERGY OF MICROPOLAR INCOMPRESSIBLE FLUIDS*

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The equations of multipolar continuum mechanics have been the subject matter of study in recent years by Eringen [1] Green [2] and others (vide the references cited in the above two papers). The linear constitutive equation for these fluids leads to an interesting theory in which micro-rotational effects and couple stress are prevalent. Such a linear theory of micropolar incompressible fluids has also been considered by Eringen [3] and Bleustein and Green [4]. In this theory the fluid motion is characterized by two vector fields \( \mathbf{V} \) and \( \mathbf{v} \) representing respectively the velocity of flow and the micro-rotation. The field equations of this theory are [3]

\[
\frac{\partial \rho}{\partial t} + \text{div} \, \rho \mathbf{V} = 0, \tag{1}
\]

\[
\rho d\mathbf{V}/dt = \rho \mathbf{f} - \text{grad} \, \rho + K \text{curl} \, \mathbf{v} - (\mu + k) \text{curl} \, \text{curl} \, \mathbf{V} + (\lambda + 2\mu + k) \text{grad} \, (\text{div} \, \mathbf{V}), \tag{2}
\]

\[
\rho d\mathbf{v}/dt = \rho \mathbf{v} + k (\text{curl} \, \mathbf{V} - 2\mathbf{v}) - \gamma \text{curl} \, \text{curl} \, \mathbf{v} + (\alpha + \beta + \gamma) \text{grad} \, (\text{div} \, \mathbf{v}). \tag{3}
\]

The constants \( \lambda, \mu, k \) are viscosity coefficients while \( \alpha, \beta, \gamma \) and the gyration parameter \( j \) are other constants of the fluid. These conform to the inequalities

\[
3\lambda + 2\mu + k \geq 0, \quad \mu \geq 0, \quad k \geq 0, \quad \gamma \geq 0, \quad |\beta| \leq \gamma, \quad 3\alpha + \beta + \gamma \geq 0 \tag{4}
\]

and it follows that

\[
\lambda + 2\mu + k \geq 0, \quad \alpha + \beta + \gamma \geq 0. \tag{5}
\]

It has been noticed by Leray, Kampé de Fériet et al. that the kinetic energy of the Navier–Stokes viscous liquid in a domain with rigid walls decays. Kampé de Fériet [5] proved that for the Navier–Stokes fluids, the decay of the kinetic energy is faster than the exponential. In the present note we obtain the corresponding result for micropolar fluids.

Let \( R \) be a domain in space bounded by the regular curve \( \Gamma \) and let the vectors \( \mathbf{V}, \mathbf{v} \) possess continuous second order derivatives in \( R \) and vanish on \( \Gamma \). The kinetic energy of the fluid is

\[
T = (\rho/2) \int V^2 \, d\tau + (\rho j/2) \int v^2 \, d\tau. \tag{6}
\]

The integrals in Eq. (6) and everywhere else in this note are over the volume of \( R \), the only exception being in Eq. (17). We have the inequalities

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\[ T_1 = (\rho/2) \int V^2 \, d\tau \leq (\rho \Lambda/8\pi) \int (\text{curl } V)^2 \, d\tau \]  

(7)

\[ T_2 = (\rho j/2) \int v^2 \, d\tau < (\rho j \Lambda/8\pi) \int \{(\text{curl } v)^2 + (\text{div } v)^2\} \, d\tau \]  

(8)

in which the constant \( \Lambda \) depends only on the geometry of the domain and is defined by

\[ \Lambda^2 = \iint (PQ)^{-2} \, d\tau \, d\tau. \]  

(9)

If the domain can be included in a ball of diameter \( d \), the constant coefficients multiplying the integrals on the right side in Eqs. (7), (8) can be replaced by \( \rho d^2/(3 + 13^{1/2})\pi^2 \) and \( \rho jd^2/6\pi^2 \) respectively, as pointed out by Serrin [6].

From Eqs. (6), (7), (8) we have

\[ T = T_1 + T_2 \leq (\rho \Lambda/8\pi) \int \{(\text{curl } V)^2 + j(\text{curl } v)^2 + j(\text{div } v)^2\} \, d\tau. \]  

(10)

Assuming that the body force is derivable from a potential field, we have

\[ \rho(\partial V/\partial t - V \times \text{curl } V) = \text{grad } F + k \text{ curl } v - (\mu + k) \text{ curl curl } V \]

and

\[ \frac{dT_1}{dt} = \int \rho V \cdot \frac{\partial V}{\partial t} \, d\tau \]

\[ = k \int V \cdot \text{curl } v \, d\tau - (\mu + k) \int V \cdot \text{curl curl } V \, d\tau \]  

(11)

\[ = k \int V \cdot \text{curl } v \, d\tau - (\mu + k) \int (\text{curl } V)^2 \, d\tau. \]

Omitting the body couple in Eq. (3), we get

\[ \rho j \frac{\partial v}{\partial t} + \rho j(V \cdot \text{grad})v = -2kv + k \text{ curl } V - \gamma \text{ curl curl } v + (\alpha + \beta + \gamma) \text{ grad } (\text{div } v). \]

From this we can see that

\[ \frac{dT_2}{dt} = \rho j \int v \cdot \frac{\partial v}{\partial t} \, d\tau = -2k \int v^2 \, d\tau \]

\[ + k \int v \cdot \text{curl } V \, d\tau - \gamma \int (\text{curl } v)^2 \, d\tau - (\alpha + \beta + \gamma) \int (\text{div } v)^2 \, d\tau. \]  

(12)

From Eqs. (11) and (12) we get

\[ \frac{d}{dt} (T_1 + T_2) = \frac{dT}{dt} = -\mu \int (\text{curl } V)^2 \, d\tau - k \int v^2 \, d\tau - \gamma \int (\text{curl } v)^2 \, d\tau \]

\[ - (\alpha + \beta + \gamma) \int (\text{div } v)^2 \, d\tau - k \int (v - \text{curl } V)^2 \, d\tau \]  

(13)

from which the decreasing nature of \( T \) is evident. Using the bounds given in Eqs. (7) and (8) for \( T_1 \) and \( T_2 \) in Eq. (13), we get
\[
\frac{dT}{dt} \leq -(8\pi\mu/\rho\Lambda)T_1 - ((2k/\rho j) + (8\pi a/\rho j\Lambda))T_2
\]
in which \(a\) is a positive number equal to \(\min(\alpha + \beta + \gamma, \gamma)\). If \(b\) denotes the positive number equal to minimum \([\mu, \{a/j + (k\Lambda/4\pi j)\}]\), we see that
\[
dT/dt \leq -(8\pi b/\rho\Lambda)T
\]
and now it follows that
\[
T(t) \leq T(t_0) \exp[-(8\pi b/\rho\Lambda)(t - t_0)].
\]
The decay of the kinetic energy is faster than the exponential rate. It would be of interest to examine if the velocity and micro-rotation also decay in this manner.

The spectral function of the kinetic energy also decreases faster than the exponential. Let \(r = (x, y, z)\) and \(\omega = (\omega_1, \omega_2, \omega_3)\) denote the position vector in the space of the fluid and in the space \(\Omega\) of the real variables \(\omega_1, \omega_2, \omega_3\). If \(V = (u, v, w)\) and \(v = (A, B, C)\) are the velocity and micro-rotation components, we define
\[
U(\omega_1, \omega_2, \omega_3, t), \quad V(\omega_1, \omega_2, \omega_3, t), \quad W(\omega_1, \omega_2, \omega_3, t),
X(\omega_1, \omega_2, \omega_3, t), \quad Y(\omega_1, \omega_2, \omega_3, t), \quad Z(\omega_1, \omega_2, \omega_3, t)
\]
to be their Fourier transforms over the domain \(R\). We have thus
\[
U(\omega_1, \omega_2, \omega_3, t) = (8\pi^3)^{-1} \int u(x, y, z, t) \exp[i(\omega \cdot r)] \, dr
\]
and the inverse relation is
\[
u(x, y, z, t) = \int U(\omega_1, \omega_2, \omega_3, t) \exp[-i(\omega \cdot r)] \, d\omega
\]
The integral in Eq. (17) is over the entire space spanned by \(\omega_1, \omega_2, \omega_3\). The spectral function \(\gamma(\omega_1, \omega_2, \omega_3, t)\) of the kinetic energy is seen to be
\[
\gamma(\omega_1, \omega_2, \omega_3, t) = 4\pi^2\rho(|U|^2 + |V|^2 + |W|^2 + j(|X|^2 + |Y|^2 + |Z|^2))
\]
From Schwarz's inequality in Eq. (16) we get
\[
(8\pi^3 |U|)^2 \leq (\text{Vol. } R) \int u^2 \, dr.
\]
From inequalities of this type for the Fourier transforms we see that
\[
\gamma(\omega_1, \omega_2, \omega_3, t) \leq (8\pi^3)^{-1}(\text{Vol. } R)T
\]
and now it is clear that
\[
\gamma(\omega_1, \omega_2, \omega_3, t) \leq (8\pi^3)^{-1}(\text{Vol. } R)T(t_0) \cdot \exp[-(8\pi b/\rho\Lambda)(t - t_0)]
\]
on using Eq. (15).

References