ON THE USE OF FOURIER TRANSFORMS FOR THE SOLUTION OF TWO-
DIMENSIONAL PROBLEMS OF ELASTOSTATICS*

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In this note we derive in a heuristic manner a condition which should be satisfied
in order that Fourier transforms may be used for the solution of two-dimensional boundary value problems in the mathematical theory of elasticity.

It is well known [1, p. 405] that the components of the displacement vector given by
the relations

\[ U_x(x, y) = \frac{1}{2(1 - \eta)} \int_0^\infty \xi^{-1}e^{-i\xi y}(1 - 2\eta - \xi |y|)\psi(\xi) \sin \xi x \, d\xi, \] (1)

\[ U_y(x, y) = \frac{1}{2(1 - \eta)} \int_0^\infty \xi^{-1}e^{-i\xi y}(2 - 2\eta + \xi |y|)\psi(\xi) \cos \xi x \, d\xi, \] (2)

where \( \psi(\xi) \) is an arbitrary integrable function, are suitable for constructing solutions of the two-dimensional problems of elastostatics, where the elastic field is symmetric about the x-axis. These solutions have the property that the shear component of the stress tensor vanishes for \( y = 0 \) and that all the components of the stress tensor and the displacement vector approach zero for large distances from the origin. The normal component of the stress-tensor for \( y = 0 \) is given by the relation

\[ \sigma_{yy}(x, 0) = -\frac{E}{2(1 - \eta)} \int_0^\infty \psi(\xi) \cos \xi x \, d\xi. \] (3)

Inherent in the use of Fourier transforms is the assumption of integrability for the components of the stress tensor and displacement vector.

Formally, since

\[ \frac{1}{\pi} \int_0^\infty \cos \xi x \, d\xi = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\xi x} \, d\xi = \delta(x), \] (4)

we obtain the relation

\[ \int_0^\infty \sigma_{yy}(x, 0) \, dx = -\frac{E}{2\pi(1 - \eta)} \psi(0). \] (5)

Since \( U_y(x, 0) \) is finite and is seen from (2) to be equal to \( \int_0^\infty \xi^{-1}\psi(\xi) \cos \xi x \, d\xi \), it is necessary that \( \psi(\xi) \) must vanish for \( \xi = 0 \), or else the integral will be divergent. It follows that

\[ \int_0^\infty \sigma_{yy}(x, 0) \, dx = 0 \] (6)

and also that Fourier transform methods are applicable only if the above condition is satisfied.

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Examples. (i) Let us consider the problem of a Griffith crack occupying the segment 
$-1 \leq x \leq 1$ in an infinite isotropic elastic sheet opened by a uniform internal pressure $p_0$. The normal component of stress is given by $-p_0$ for the crack surface and for $x > 1$ 
(see [1]), $\sigma_{yy}(x, 0) = -p_0(1 - x/(x^2 - 1)^{1/2})$ and it is easily verified that $\int_0^\infty \sigma_{yy}(x, 0) \, dx = 0$. Physically, this means that the algebraic sum of the normal load transmitted to the medium across the plane $y = 0$ is zero.

(ii) As a second example, we consider the indentation problem of the semispace by frictionless punch which produces a specified even displacement for $-1 \leq x \leq 1$, it being assumed that the region on the boundary not immediately under the punch (i.e. $|x| > 1$) is stress-free. In this case, the auxiliary function $\psi(\xi)$ is to be determined by the pair of equations

$$\int_0^\xi \xi^{-1} \psi(\xi) \cos \xi x \, d\xi = f(x), \quad 0 \leq x \leq 1 \quad (7)$$

$$\int_0^\infty \psi(\xi) \cos \xi x \, d\xi = 0, \quad x > 1. \quad (8)$$

Recall that

$$\sigma_{yy}(x, 0) = \int_0^\infty \psi(\xi) \cos \xi x \, d\xi$$

and in view of (8), we find on integrating the above equation from $t$ to $\infty$, that

$$\int_t^1 \sigma_{yy}(x, 0) \, dx = \int_0^\infty \frac{\psi(\xi)}{\xi} \sin \xi t \, d\xi, \quad 0 \leq t < 1. \quad (9)$$

Since $\psi(\xi)$ is integrable and vanishes for $\xi = 0$, it is obvious that

$$\int_0^1 \sigma_{yy}(x, 0) \, dx = \int_0^\infty \sigma_{yy}(x, 0) \, dx = 0.$$

It is also implied that the normal stress is both compressive and tensile under the punch, a fact which is corroborated by the relation

$$\int_0^1 f(x) \, dx \frac{1}{\sqrt{(1 - x^2)}} = 0 \quad (10)$$

obtained by Sneddon [2, p. 101] and Lowengrub [3, p. 71] for the existence of the solution of (7) and (8).

While for the sake of convenience we have considered the transfer of load across the plane $y = 0$, it can be concluded that across any plane $y = k$ the sum of the load transferred to the medium $y > k$ is equal to zero.

References