ON CONVECTION AND DIFFUSION

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1. Preliminaries. In a recent paper [1], Marris and Passman developed the theory of the transport of a general solenoidal vector in the motion of a continuum. This constituted a generalization of part of the elegant classical theory of vorticity transport, as set forth by Truesdell [2]. I show here that part of the work of Marris and Passman can be considered to be a special case of a somewhat more general theory, and give another application of that theory. With certain minor exceptions, I use the notations and assumptions of Truesdell and Toupin [3].

2. The general integral formula. Let \( \mathbf{J} \) be any twice continuously differentiable vector function with covariant components \( \beta_k \). Form the material expression \( \beta_k \cdot x^k, \beta \). Then

\[
\frac{d}{dt} (\beta_k \cdot x^k, \beta) = \dot{\beta}_k \cdot x^k, \beta + \beta_k \cdot \dot{x}^k, \beta .
\]

Thus

\[
\frac{d}{dt} (\beta_k \cdot x^k, \beta) = \dot{\beta}_k \cdot x^k, \beta + \beta_k \cdot \dot{x}^k, \beta .
\]

Let \( \beta_a, \beta \) represent the values of \( \beta_k \) at \( t = 0 \). The integration of (2) along the path of a particle yields

\[
\beta_k \cdot x^k, \beta = \beta_a, \beta + \int_0^t (\dot{\beta}_k \cdot x^k, \beta + \beta_k \cdot \dot{x}^k, \beta) \, dt,
\]

which, with appropriate changes, yields

\[
\beta_k = \left[ \beta_a, \beta + \int_0^t (\dot{\beta}_k \cdot x^k, \beta + \beta_k \cdot \dot{x}^k, \beta) \, dt \right] X^a, l X^b, l .
\]

Eq. (4) is a relationship for the transport of the gradient of an arbitrary vector in the motion of a continuum. The first formula of this type was published in 1948 by Truesdell [5].

It is seen that there are two processes involved in the transport of the gradient of \( \mathbf{J} \). One, represented here by the term

\[
\beta_{\beta, a} X^a, k X^k, l ,
\]

depends only on the value of \( \mathbf{J} \) at \( t = 0 \) and the initial and final coordinates of the
particle, and is called convection. The second process, here represented by the term
\[
\left[ \int_0^t \left( \beta_{i,j} + \beta_{p,i} \hat{x}^p_{i,j} \right) x^j_{i,a} x^i_{j,\beta} \, dt \right] X^a_{i,k} X^\beta_{j,i}
\]
depends on the motion and the values of $\beta$ between 0 and $t$. It is called diffusion.

The axial-vector of (4) yields the relation (2.11) of Marris and Passman [1] which, in turn, yields relations for the transport of $N$th-order vorticity, including Truesdell’s formula for transport of first-order vorticity. The symmetric part of (4) with the substitution $\beta_k = \hat{x}_k$ yields the formula for transport of stretching ([3], p. 381).

3. An application. The covariant components of the $N$th Rivlin-Ericksen tensor are given by
\[
A_{pq}^{(N)} = 2 d_{pq}^{(N)} + \sum_{K=1}^{N-1} \binom{N}{K} x_m x_n x_p x_q,
\]
where $x^k$ are the contravariant components of the $N - 1$st acceleration, defined by
\[
x^{(N)} x^k = \frac{\partial x^k}{\partial t^{(N)}},
\]
and $d_{pq}^{(N)}$ are the covariant components of the $N$th stretching tensor, given by
\[
d_{pq}^{(N)} = x_{(k,m)}^{(N)}.
\]

The symmetric part of (4) is
\[
\beta_{(k,l)} = \left[ \beta_{(\beta,\alpha)} + \int_0^t \left( \hat{\beta}_{(i,j)} + \beta_{k,(i)} \hat{x}^p_{i,j} \right) x^j_{i,a} x^i_{j,\beta} \, dt \right] X^a_{i,k} X^\beta_{j,k}.
\]

We thus have the following relation for transport of the $N$th Rivlin-Ericksen tensor:
\[
A_{pq}^{(N)} = 2 \left[ d_{pq}^{(N)} + \int_0^t \left( d_{ij}^{(N+1)} + d_{ij}^{(N)} \hat{x}^p_{i,j} \right) x^j_{i,a} x^i_{j,\beta} \, dt \right] X^a_{i,k} X^\beta_{j,q}
\]
\[
+ \sum_{K=1}^{N-1} \binom{N}{K} \left[ \left( \binom{N}{K} x_{\beta,a} + \int_0^t \left( \binom{N}{K-1} x_{i,j} + \binom{N}{K} x_{r,i} \hat{x}^r_{i} \right) x^j_{i,a} x^i_{j,\beta} \, dt \right) X^a_{m,k} X^\beta_{n,q} \right.
\]
\[
\left. \cdot \left( \binom{N}{K} x_{k,\gamma} + \int_0^t \left( \binom{N}{K+1} x_{k,n} + \binom{N}{K} x_{r,n} \hat{x}^r_{k} \right) x^k_{\gamma,k} x^k_{j,q} \, dt \right) X^\gamma_{j,k} \right].
\]

It might be speculated that there exists a relation for the transport of a Rivlin-Ericksen tensor which depends only on other Rivlin-Ericksen tensors and the second referential kinestate. I have not been able to derive such a relation. Eq. (9) indicates that the $N$th Rivlin-Ericksen tensor at time $t$ is, in general, the result of a much more complex transport mechanism. It is also seen that there exist conditions under which the transport of a Rivlin-Ericksen tensor is due to convection only. It would be interesting to study these conditions.

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References