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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi x/2b)}{\cos (\pi a/2b)} \text{ is preferable to } \frac{\cos \dfrac{\pi x}{2b}}{\cos \dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\textstyle\int u^{-1} \sin u \, du \text{ is preferable to } \displaystyle\int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten.*

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

BOOK REVIEW SECTION

Differential geometry. By J. J. Stoker. John Wiley and Sons, Inc., New York, London, Sydney, Toronto, 1969. xxi + 404 pp. $14.95.

The titles of the ten chapters in this book indicate its scope: I: Operations with vectors. II: Plane curves. III: Space curves. IV: The basic elements of surface theory. V: Some special surfaces. VI: The partial differential equations of surface theory. VII: Inner differential geometry in the small from the extrinsic point of view. VIII: Differential geometry in the large. IX: Intrinsic differential geometry of manifolds. Relativity. X: The wedge product and the exterior derivative of differential forms, with applications to surface theory. There are two appendices (A: Tensor algebra in affine, Euclidean, and Minkowski spaces. B: Differential equations) and also a bibliography and an index. At the ends of the chapters there are sets of problems, including some new ones.

The book is excellently produced, clearly printed and with many diagrams. However it appears that in Chern's theorem (p. 148), $>$ should be replaced by $\geq$, and I noticed trivial misprints on pp. 152, 160, 208, 292, 320, 376, 390.

A most attractive feature of this book is the feeling of leisure it imparts. This is mathematics in the grand style. Professor Stoker is not in a hurry. He modestly describes himself as an amateur, but that is to be taken with a grain of salt, unless interpreted in the sense that he loves differential geometry and is prepared to devote much care to revealing the motivation underlying formal arguments. In the introduction he sets out his objectives. They are, first, to write a thorough but elementary treatment of differential geometry for young students, including (the second objective) a treatment of a rather large number of problems of differential geometry in the large, and, thirdly, to introduce and use three different notations—vectors, tensors, and invariant differential forms.

Differential geometry is essentially the application of the calculus to the study of geometric forms. Its most primitive achievement is to tell us that dy/dx is the slope of the tangent to a plane curve, and (from the standpoint of a physicist) its highest achievement to date is to equate gravitation to the Riemannian geometry of a 4-space. The peculiar fascination of the subject seems to lie in its Janus-character, one face looking back with nostalgia to simple pictures and the other staring unflinchingly at massive formulae. Professor Stoker has been at great pains to look both ways, as the many diagrams testify, but these unfortunately, perhaps inevitably, peter out towards the end of the book—differential forms, exterior products and exterior differentiation do not seem to lend themselves to graphic illustration.

What is a Riemannian manifold? Is the line-element positive-definite or not? On p. 207 Professor Stoker declares in favour of positive-definite, but with a footnote stating that this will be relaxed later in dealing with relativity. I do not like to see the indefinite line-element of space-time treated in this cavalier way by a geometer. It is true that *formally* it makes little difference (e.g. the Riemann tensor has the same form in each case), but *geometrically* a Euclidean 4-space and space-time are very different animals (no null lines in Euclidean 4-space, but in space-time perhaps the most interesting and important curves there are). On the whole I think it would have been better to stick to the positive-definite line-element and let relativity go to pot as far as this book is concerned, for in the short space allotted to relativity (pp. 318–333) it was really impossible to deal with it except along conventional historical lines. This is such a small part of an otherwise excellent book that it may be inappropriate to dwell on it, but I cannot refrain from blowing off a little steam as follows.

I do not agree that (p. 318) the general theory of relativity can be explained only on the basis of the special theory; this is a dangerous suggestion, for the special theory deals with frames of reference, and in the general theory there are none unless you build them to your own specification. I do not agree that (p. 319) a fixed coordinate system in the universe simply cannot be conceived. Newton conceived it, and so do you and I when we are being Newtonian (as most physicists are most of the time). There is something wrong with ds^2 on pp. 327–330, since the g_{ik} of p. 330 make the ds of p. 327 imaginary for the world line of a particle. It is a mistake to suppose that (p. 327) the gravitational field can be transformed away at a point—the Riemann tensor remains, and it *is* the gravitational field. As for Einstein's general field theories (p. 329), it is over-generous to say that they were even *partially* successful.

In a matter as complicated as the calculation of the Newtonian advance of Mercury's perihelion, it would be nice to have a reference to the source of the quoted figure.

My verdict, then, in a nutshell: an excellent book for positive-definite line-elements (and that is nearly all of it), but not so good for indefinite line-elements (as required for relativity).

J. L. SYNGE (*Dublin*)

The chi-squared distribution. By H. O. Lancaster. John Wiley and Sons, Inc., New York, 1969. xiv + 356 pp. $14.95.

Most books on statistics nowadays are texts and go through those parts of the subject that fall out as a special case of some general principle. The finest example of this is surely Lehmann's *Testing statistical hypotheses.* But as one goes down the list, these books get very souless and omit much that is of importance. One topic which is central to both theory and practice which usually gets neglected is the entire collection of topics related to chi-square test. Lancaster, apart from a quite distinct and distinguished career as a public-health statistician, has devoted all his life to this area. Thus instead of a brief view of chi-square as a special case of statistical theory, we can see here the history of statistical ideas that have any bearing on chi-square, and the associated mathematics. It seems to this reviewer that the atmosphere is healthier when practical sense and mathematical naturalness are followed instead of the current "party line"—perhaps mostly because all the fascinating mathematical side avenues and cross connections are mentioned. The exercises and complements are quite rich too.

If a sample of n objects is drawn from an infinitely large population which has a fraction p_i of class $i (i = 1, \cdots, k)$, one would expect to see n_i objects from class i where n_i is approximately equal to $np_i (i = 1, \cdots, k)$. Thus a reasonable measure of the unusualness of sample $(n_1, n_2, \cdots, n_k)$ where $\sum_1^k n_i = n$ is given by

$$X^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i}. \tag{1}$$

This is the basic form of Pearson's chi-square statistic. Since the probability of every possible sample is given by the appropriate term of the multinomial $(p_1 + \cdots + p_k)^n$, one has to add the probabilities of all samples giving X^2 the same value y to find $\operatorname{Prob}\{X^2 = y\}$. When this is done for every one of the finite set of possible values of X^2, one has the probability distribution of X^2. It is an astonishing fact that this discrete distribution is, in practice, very well approximated by a continuous distribution depending only on k—the so-called χ^2 distribution with $k - 1$ degrees of freedom. That is,

$$\operatorname{Prob}(X^2 \leq y) \approx \left[2^{k-1/2}\Gamma\left(\frac{k-1}{2}\right)\right]^{-1} \int_0^y x^{k-1/2-1} e^{-x/2}\, dx. \tag{2}$$

The proof follows from a central limit theorem and the reduction of (1) to be a sum of squares of $k - 1$ independent normal variables. When the sample is drawn from a multinomial $(\pi_1 + \pi_2 + \cdots + \pi_k)^n$, the distribution of (1) is approximated by what is called the non-central χ^2-distribution depending now on k and a simple function of the p_i's and π_i's. The proofs, merits and uses of these approximations are the subject matter of Chapters 2, 3, 5, 7. Chapter 1 is an historical survey including pages of fascinating quotations. The form of X^2 suggests that orthogonal functions will be relevant and these are discussed in Chapter 4.

Very often the p_i are known functions of an unknown parameter θ or $p_i = p_i(\theta)$, $i = 1, \cdots, k$. Chapter 8 describes the several asymptotically equivalent ways of handling this case which gives a result that is so elegant and makes chi-square so attractive in practice. It is almost certain that Fisher evolved some of his most fundamental new ideas while wrestling with this problem.

Chapter 9 deals theoretically with many points which arise in practice. The relevant theory is rarely given—just the consequences. In such a definitive book perhaps this is a pity. The bibliography of 54 pages is obviously complete but some important papers there receive no discussion. Many will feel that Hoeffding's work should have received far more than just the reference. The information theory approach too might have been mentioned, although again Kullback is referred to. Is it too much to expect that a chi-square book would discuss its competitors?

Chapters 10, 11 and 12 deal with contingency tables. The prototype is the 2 × 2 table giving, e.g., the numbers of inoculated, and not inoculated, people who did, or did not, get some illness. One is interested in whether inoculation is a good thing or not. While this type of problem may be thrown into the earlier multinomial format, one's intuition is then sometimes lost. Many classifications are explicitly based on continuous variables. If the parallel with the independence of random variables is retained, then many new developments follow. There are thus connections with multivariate analysis to be exploited.

The book concludes with a chapter-by-chapter index to the vast bibliography and with a subject index.

All libraries should certainly get this book. As it is orthogonal to all but one modern text, it would be great fun to use it for a special topics course—everyone trained after 1950 is almost sure to see another aspect of statistics so both staff and students should enjoy it! In conclusion, I now think the ideal review would be a series of quotes of non-obvious material—but this would not be appropriate for a mathematical journal, most of whose readers have never heard of Pearson's chi-square.

G. S. WATSON (*Princeton, N. J.*)

Nicht-numerische Informationsverarbeigung. By R. Gunzenhaüser. Springer-Verlag, Wien, New York, 1969. xx + 509 pp. $29.50.

From the cover:

> "Since the beginnings of the development of non-numerical information processing no comprehensive treatment of this field has appeared in the literature. Therefore a team of experts has undertaken the task of describing the field from various points of view. Every author reports on his speciality.
>
> For the reader who is unfamiliar with the methods and the typical problems of the areas of non-numerical data processing by computer, Professor Knödel has written an easily readable, informative introduction. Thereafter more than twenty subject areas of non-numerical information processing are surveyed, all of which are central to current research activities. Prospects and limitations of further research are indicated...
>
> ... It is the authors' intention to form opinions, not by preaching of certain gospels, but rather through the dissemination of comprehensive scientific knowledge."

The twenty-seven authors' first and prime mistake was to choose a far too ambitious objective, and this can be used as a general excuse for their other failures which essentially amount to missing their stated goals. There is one implicit fact, although not a stated objective, which is conveyed in convincing clarity: non-numerical data processing at present consists of a wide variety of problems for solution of which the digital computer has been employed, either commercially, or experimentally, or for fun. It is a conglomeration of case-studies badly in need of an underlying theoretical or technical foundation to display their common aspects. The present book enumerates and describes various computer application areas; but the reader who expects an exposition of theories and techniques commonly applicable to these problems is going to be disappointed. This disappointment starts with the introduction, which rushes the reader from the era of the desk calculator to that of time-sharing and modularity of third-generation computer systems, through discussions of information encoding, logical operations, storage media, and programming languages. And all this within thirteen pages! The result: the reader who starts as an expert is bored, the one who is a layman remains a layman, possibly with the augmented misconception of having gained deeper insight into a new subject.

The large remainder of the book covers a wide variety of problems. They range from random number generators to conversion algorithms of Roman to Arabic numerals, from numerical quadrature to algebraic formula manipulation, from compiling of algorithmic languages to compiler-compilers, from chess-playing computers to simulation techniques, from automation of production assembly lines to PERT planning, from school timetable generation to analysis of natural languages by computer. It includes, moreover, chapters on mechanical language translation, automatic documentation systems, data processing in libraries, pattern recognition, sorting of letters in the post office, generation of aesthetic objects by computer, and ends with a section on computer-aided instruction. The table of contents alone takes up ten pages.

This reviewer is, alas, no expert in most of these subjects, but he found the promises of the authors not satisfactorily met in those subjects in which he feels he has some competence, and he is of course tempted to extrapolate. For instance, the section on formula translation and compilers is too sketchy, and to a large extent outdated. The bibliography contains, typically, 46 references, most of which are only of interest to the historian. The subject of formula manipulation or symbolic computation is covered in only three pages, including a sketch of FORMAC. On the other hand, more than twenty pages are dedicated to the problem of postal letter sorting, and equally many to the generation of "aesthetic objects."

The general impression is that too little is said about far too many topics which are chosen arbitrarily from the wide world of existing problems. Their individual aspects are presented, but a discussion of techniques or theories useful in the solution of the problems mentioned as well as future problems is missing. And this is precisely what one would expect from a useful book about data processing, numerical or non-numerical. The book is certainly not worth its price; it suffices if it is available in the libraries of scientific institutions. And such libraries will buy it anyway.

N. WIRTH (*Zürich*)