A GENERAL ONE-PHASE STEFAN PROBLEM*

BY

B. SHERMAN

Rocketdyne, a Division of North American Rockwell Corporation, Canoga Park, California

1. Introduction. In this paper we discuss the problem of determining $u(x, t)$ and the free boundary $s(t) > 0$ subject to the conditions

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= u_t, \quad 0 < x < s(t); \quad u(x, 0) = \varphi(x), \quad u_t(0, t) = f(t), \\
\frac{\partial u}{\partial x}(s(t), t) &= g(t), \quad -\lambda(0) s'(t) + u_t(s(t), t) = q(t), \quad s(0) = a.
\end{align*}
\]

Here $f$, $g$, $\lambda$, $q$ are defined for all $t \geq 0$, and $\varphi$ and $a$ are the given data of the problem, with $\varphi(a) = g(0)$, $a > 0$, and $\lambda(t) > 0$. When

\[
g(t) \equiv 0, \quad q(t) \equiv 0, \quad \lambda(t) = \lambda = \text{constant}, \quad \varphi(x) \leq 0, \quad f(t) \geq 0,
\]

then (1.1) has a simple interpretation as a problem in heat conduction with melting: the region $0 \leq x \leq a$ is initially solid with the temperature distribution $\varphi(x)$, the region $a < x < \infty$ is liquid at the melting temperature 0, there is flux $f(t)$ directed out of the solid at the fixed face $x = 0$, and $\lambda = \rho l / k$ where $\rho$ is the common density of liquid and solid, $l$ is the latent heat, and $k$ the coefficient of thermal conductivity. $u(x, t)$ is the temperature in the solid part and $s(t)$ the position of the interface. With the restrictions (1.2) on the data it has been proved (when regularity conditions are imposed on the data) that $u(x, t)$ and $s(t)$ exist for all $t$, are unique, depend continuously on the data, and that $s'(t) > 0$ [2], [3], [5], [7], [8]. In these papers the region $0 \leq x \leq a$ is liquid and $a < x < \infty$ is solid so the inequalities on $\varphi$ and $f$ are reversed and $-\lambda$ is replaced by $\lambda$ in the free boundary condition. The case $a = 0$, subject to (1.2), is treated in [2], [3], [5], [12].

If in (1.2) we remove the restriction $q(t) \equiv 0$ and allow $q(t)$ to be an arbitrary continuous function

\[
g(t) \equiv 0, \quad \lambda(t) = \lambda = \text{constant}, \quad \varphi(x) \leq 0, \quad f(t) \geq 0,
\]

then existence, uniqueness, and continuous dependence are proved in [10], [11] when regularity conditions are imposed on the data. The condition $q(t) \geq 0$ is imposed in [10] but this restriction is unnecessary and plays no role in the proofs. The interpretation of (1.1) as a heat conduction problem with melting is as above, with $q(t)$ a flux situated in the moving interface which may be directed into ($q(t) > 0$) or out of ($q(t) < 0$) the solid. The free boundary $s(t)$ is not, in general, a monotonic function. Complete melting may occur in a finite time $\sigma$, where $\sigma$ is the smallest value satisfying $\lambda a + H(\sigma) = 0$, where

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\[ H(t) = \int_0^t [f(\tau) - q(\tau)] \, d\tau - \int_0^a \varphi(x) \, dx. \] (1.4)

If there is no \( t \) such that \( \lambda t + H(t) = 0 \) then complete melting does not occur in a finite time; we then write \( \sigma = \infty \). Thus the solution of (1.1) exists in the case (1.3) to \( t = \sigma \).

The existence and uniqueness proof for (1.1) in the case (1.3) is divided into two parts, first a proof of local existence and uniqueness, and then the extension of that solution to \( t = \sigma \). The proof of the local theorem does not make use of the hypotheses on the signs of \( f(t) \) and \( \varphi(x) \); only the regularity properties of these functions are used. Accordingly it seems plausible that, with appropriate regularity conditions in the data, we can prove a local existence and uniqueness theorem for (1.1) with no restrictions on the signs of the data and with the requirement \( g(t) = 0 \) removed. This is done in Sec. 2. Next we define \( \sigma \) to be the supremum of those \( t \) for which (1.1) has a solution. For (1.2) we know that \( \sigma = \infty \) and for (1.3) we know that \( \sigma = \infty \) or \( s(\sigma) = 0 \) with \( s(t) > 0 \) for \( 0 < t < \sigma \). The question then arises as to whether, for (1.1), it is possible for \( \sigma \) to be finite and for \( s(t) \) not to approach 0 as \( t \to \sigma \). We show in Sec. 2 that there is indeed this third possibility and that for such a \( \sigma \) \( \lim \inf u(\sigma(t), t) = -\infty \) as \( t \to \sigma \). It follows that \( \lim \inf s'(t) = -\infty \) as \( t \to \sigma \). We show by an example that this third possibility does occur. These three possibilities describe completely the possible behavior at \( t = \sigma \).

If in (1.1) we delete the term \(-\lambda s'(t)\) then we get a free boundary problem for the heat equation in which Cauchy data are prescribed on the free boundary. Wentzel [14] has discussed a problem of this sort. The following problem arises in statistical decision theory:

\[ u_{xx} = u_t, \quad 0 < x < s(t); \quad u_x(0, t) = 1/2, \quad s(0) = 0, \]

\[ u(s(t), t) = 1/2t, \quad u_x(s(t), t) = 0. \] (1.5)

The problem discussed by Chernoff in [1] can be reduced to (1.5). We may consider the possibility of obtaining existence theorems for this general class of problems by adding the term \(-\lambda s'(t)\), \( \lambda \) a parameter, to the flux condition at the free boundary and then letting \( \lambda \to 0 \). We have elsewhere [13] obtained some results in this direction.

2. Existence and uniqueness of a solution of (1.1). We will suppose the data of (1.1) satisfy the following regularity conditions: \( f(t), g'(t), \lambda(t) \) are continuous on \( t \geq 0 \) and \( \varphi'(x) \) is continuous on \( 0 \leq x \leq a \). We suppose that \( \lambda(t) > 0 \) and that \( \varphi(a) = g(0) \). The regularity conditions are not minimal but they are appropriate for the fixed point method of proof we use here. We note again that \( a > 0 \).

We define a solution \( u(x, t), s(t) > 0 \) of (1.1) for \( 0 \leq t < T \), where \( 0 < T \leq \infty \), as follows [6, p. 216]: (a) \( u_{xx} \) and \( u_t \) are continuous for \( 0 < x < s(t), 0 < t < T \); (b) \( u \) and \( u_x \) are continuous for \( 0 \leq x \leq s(t), 0 < t < T \); (c) \( u \) is continuous also for \( t = 0, 0 < x \leq a \) and \( u \) is bounded at \((0, 0)\); (d) \( s'(t) \) exists and is continuous on \( 0 \leq t < T \); (e) (1.1) is satisfied. To define a solution of (1.1) for \( 0 \leq t \leq T \) the \( < T \) inequalities in (a), (b), (d), and (e) are replaced by \( \leq T \).

**Theorem.** For sufficiently small \( T \) (1.1) has a unique solution. Let \( \sigma \) be the supremum of those values of \( T \) for which (1.1) has a solution. Then the solution for \( 0 \leq t < \sigma \) is unique and there are three possibilities: (a) \( \sigma = \infty \); (b) \( \sigma \) is finite and \( s(\sigma -) = 0 \); (c) \( \sigma \) is finite, \( \lim \inf u_x(s(t), t) = -\infty \) as \( t \to \sigma \), and \( s(t) \) does not tend to 0 as \( t \to \sigma \).

To prove the theorem we introduce the fundamental solution of the heat equation, \( K \),
and the Green and Neumann functions $G$ and $N$ for the upper half plane $t > 0$ [6]:

\[ K(x, t; x', t') = [4\pi(t - t')]^{-1/2} \exp \left[ -(x - x')^2/4(t - t') \right], \]

\[ G(x, t; x', t') = K(x, t; x', t') - K(x', t; x, t'), \]

\[ N(x, t; x', t') = K(x, t; x', t') + K(x', t; x, t'). \]

Let $u(x, t), s(t)$ be a solution of (1.1) for $0 \leq t \leq T$. Integrating the identity

\[ \frac{\partial}{\partial \xi} (Nu_t - uN_t) - \frac{\partial}{\partial \tau} (Nu) = 0 \]

over the domain $0 < \xi < s(\tau), \epsilon < \tau < t - \epsilon$, using Green's theorem, and letting $\epsilon \to 0$, we get

\[ u(x, t) = \int_0^t \left[ u_i(s(\tau), \tau) + g(\tau)s'(\tau) \right] N(x, t; s(\tau), \tau) d\tau \]

\[ - \int_0^t g(\tau)N_t(x, t; s(\tau), \tau) d\tau \]

\[ - x(t) N(x, t; 0, \tau) d\tau + \int_0^\alpha \varphi(\xi)N(x, t; \xi, 0) d\xi. \quad (2.1) \]

In (2.1) we replace $N_t$ by $-G_x$ and then differentiate with respect to $x$ on both sides. The $G_x$ which appears in the second term on the right may be replaced by $G_x = -G_x$. We have

\[ \frac{d}{d\tau} G(x, t; s(\tau), \tau) = G_t(x, t; s(\tau), \tau)s'(\tau) + G_t(x, t; s(\tau), \tau). \quad (2.2) \]

Using (2.2) and replacing $N_x$ by $-G_t$ in the last term on the right (in the differentiated version of (2.1)) we get, on performing partial integrations, using $\varphi(\alpha) = g(0)$, and writing $v(t) = u_s(s(t), t)$

\[ u_s(x, t) = \int_0^t v(\tau)N_x(x, t; s(\tau), \tau) d\tau - \int_0^t g(\tau)G_t(x, t; s(\tau), \tau) d\tau \]

\[ - x(t) N(x, t; 0, \tau) d\tau + \int_0^\alpha \varphi(\xi)G(x, t; \xi, 0) d\xi. \quad (2.3) \]

In (2.3) we let $x \to s(t)$. Using Lemma 1 of [4] we get

\[ v(t) = 2 \int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau) d\tau + 2 \int_0^t g(\tau)G(s(t), t; s(\tau), \tau) d\tau \]

\[ - 2 \int_0^t f(\tau)N_x(s(t), t; 0, \tau) d\tau + 2 \int_0^\alpha \varphi(\xi)G(s(t), t; \xi, 0) d\xi. \quad (2.4) \]

We have also the equation

\[ s(t) = a - \int_0^t (\lambda(\tau))^{-1} g(\tau) d\tau + \int_0^t (\lambda(\tau))^{-1} v(\tau) d\tau. \quad (2.5) \]

We must show now that a solution $v(t)$ and $s(t) > 0$ of (2.4) and (2.5) continuous on $0 \leq t \leq T$ gives a solution of (1.7) by defining $u(x, t)$ by (2.1), with $u_t(s(\tau), \tau)$ replaced by $v(\tau)$. The proof of this parallels the argument in [10, p. 57] except that in equation (12) of [10] must replace $u(s(\tau), \tau)$ by $u(s(\tau), \tau) - g(\tau)$.

To prove the existence of a solution of (2.4) and (2.5) let $C(T)$ be the Banach space of continuous functions on $0 \leq t \leq T$ with maximum norm. Let $C(T, M)$ be the closed sphere $\|v\| \leq M$. If we write $v = Sv$ for (2.4) then $w = Sv$, with $s(t)$ defined by (2.5), defines a
mapping of $C(T, M)$ into $C(T)$. We show that we can choose $T$ and $M$ so that $S$ is a contracting mapping of $C(T, M)$ into itself. Following the argument in [10, p. 57-60] we see that $S$ is a mapping of $C(T, M)$ into itself if $M = |\varphi'(a)| + 1$ and $T$ is subject to the inequalities (16) and (20) of [10], where in (20) we must insert the term $2 ||q'|| T$ on the left (also $k = 1$ and $\alpha = ||\lambda^{-1}||$ in those inequalities). If, furthermore, inequalities (28) of [10] and $F(M, T) + ||q'||(M, T) < \frac{1}{2}$ are satisfied, where $F(M, T)$ is defined in [10, p. 60] and $f(M, T)$ goes to 0 with $T$, then $S$ is a contracting mapping of $C(T, M)$ into itself. Thus there is a unique fixed point, i.e., (2.4) and (2.5) have solutions on $0 \leq t \leq T$ which is unique subject to $||\varphi|| \leq M$. That the uniqueness is independent of this condition is proved as in [10, p. 61].

We have proved above the existence of a local solution of (1.1), i.e., that $\sigma > 0$. That the solution of (1.1) is unique on $0 \leq t < \sigma$ is proved the same way as uniqueness for the local solution. To prove the final part of the theorem we note there are the following possibilities, if $\sigma$ is finite, for $v(t)$ as $t \to \sigma$:

(a) $v(t)$ tends to a finite limit,

(b) $v(t) \to + \infty$,

(c) $\lim \inf v(t)$ is finite, $\lim \sup v(t) = + \infty$, (2.6)

(d) $v(t)$ is bounded,

(e) $\lim \inf v(t) = - \infty$.

Throughout the ensuing discussion we may assume that, if $s(t)$ has a finite limit, that limit is positive, for otherwise we are in case (b) of the theorem. Suppose (2.6a) is true. Then $s(t)$ also has a finite limit as $t \to \sigma$. For $0 < t < \sigma$ we may consider the inequalities specified in the previous paragraph, where $\alpha = s(t), \varphi(x) = u(x, t), M(t) = |w(t)| + 1$, and norms of functions of $t$ are taken over the interval from $t$ to $t + T$. Let $T^*(t)$ be the supremum of those $T(t)$ satisfying the inequalities. Then $T^*(t) > 0$ and, since $v(\sigma-) \exists$, $T^*(\sigma-) \exists$ and is positive. We may then choose $t$ sufficiently close to $\sigma$ so that $T^*(t) > \sigma - t$. This implies that the solution of (1.1) can be extended past $\sigma$, a contradiction. Thus (2.6a) cannot be true. (2.6b) implies that $v(t)$ is positive in the vicinity of $\sigma$ and therefore bounded (by the argument in [10, p. 62]), a contradiction. (2.6d) implies, again by the argument in [10, p. 62], that $v(t)$ has a finite limit as $t \to \sigma$ so that the solution can be extended past $\sigma$; thus (2.6d) is also ruled out. It remains to eliminate (2.6c). Let $m = \inf v(t)$ on $0 \leq t \leq \sigma$ and let $v^*(t) = v(t) - m, q^*(t) = q(t) - m$. Then $v^*(t) \geq 0$. From

$$s(t) = a - \int_0^t (\lambda(\tau))^{-1} q^*(\tau) \, d\tau + \int_0^t (\lambda(\tau))^{-1} v^*(\tau) \, d\tau$$

we see that either $s(\sigma-) \exists$ or $s(\sigma-) = + \infty$. Analogous to (2.4) we may write, where $\mu < \sigma$ is to be determined and $\sigma - \mu \leq t < \sigma$,

$$v^*(t) = -m + 2 \int_{\sigma - \mu}^t v^*(\tau) N_x(s(t), t; s(\tau), \tau) \, d\tau + 2m \int_{\sigma - \mu}^t N_x(s(t), t; s(\tau), \tau) \, d\tau$$

$$+ 2 \int_{\sigma - \mu}^t g'(\tau) G(s(t), t; s(\tau), \tau) \, d\tau - 2 \int_{\sigma - \mu}^t f(\tau) N_x(s(t), t; 0, \tau) \, d\tau$$

$$+ 2 \int_0^s u(\xi, \sigma - \mu) G(s(t), t; \xi, \sigma - \mu) \, d\xi. \quad (2.7)$$

The second term, the last, and the next to last term on the right of (2.7) may be estimated by the methods of [10, p. 62]. Let $||v^*||$ and $||u^*||$ be respectively the maximum on $\sigma - \mu \leq \tau \leq t$ of $v^*(\tau)$ and on $0 \leq x \leq s(\sigma - \mu)$ of $|u_x(x, \sigma - \mu)|$. The norms of other functions of $t$ will be taken over $0 \leq t \leq \sigma$. Then the second term is $\leq \mu^{1/2} ||\lambda^{-1}|| ||v^*|| ||q^*||$, the last
term is \( <41|«,||, and the next to last term is \( <||/||. The term involving \( g'(\tau) \) is \( <4\mu^{1/2} ||g'||. \)

The third term on the right of (2.7) is the sum of two terms, of which the first is

\[
m \int_{\tau - \mu}^{\tau} \left[-(s(t) - s(\tau))/(t - \tau)\right] K(s(t), t; s(\tau), \tau) \, d\tau
\]

\[
= m \int_{\tau - \mu}^{\tau} \left[\int_{\tau}^{t} \lambda^{-1}(\xi)q^*(\xi) \, d\xi - \int_{\tau}^{t} \lambda^{-1}(\xi)q^*(\xi) \, d\xi\right] (t - \tau)^{-1} K(s(t), t; s(\tau), \tau) \, d\tau
\]

\[
\leq \mu^{1/2} \left|m\right| \left|\lambda^{-1}\right| \left(||q^*|| + ||v^*||\right),
\]

and the second is

\[
\left(m/2\pi^{1/2}\right) \int_{\tau - \mu}^{\tau} -\left(s(t) + s(\tau)/(t - \tau)^{3/2}\right) \exp \left[-(s(t) + s(\tau))^{2}/4(t - \tau)\right] \, d\tau. \tag{2.8}
\]

If \( m \geq 0 \) then (2.8) has the upper bound 0. If \( m < 0 \) and \( s(t) \) has a finite positive limit at \( \sigma \) then, letting \( a(\mu) \) and \( b(\mu) \) be, respectively, the maximum and minimum of \( s(\tau) \) on \( \sigma - \mu \leq t \leq \sigma \), (2.8) is less than or equal to

\[
|m| \pi^{-1/2} \int_{\tau - \mu}^{\tau} a(t - \tau)^{-3/2} \exp \left(-b^2/(t - \tau)\right) \, d\tau
\]

\[
= 2 \left|m\right| a/b \pi^{1/2} \int_{b/(t - \tau + \mu)^{-1/2}}^{\infty} \exp \left(-\xi^2\right) \, d\xi < \left|m\right| a/b.
\]

Here \( b(\mu) > 0 \). If \( m < 0 \) and \( s(t) \to +\infty \) as \( t \to \sigma \) then for sufficiently small \( \mu \) \( s(t) + s(\tau) < (s(t) + s(\tau))^{2} \) and, using \( xe^{-x^2} \leq e^{-1/x^2} \) for \( x > 0 \), (2.8) is less than

\[
2 \left|m\right|/\pi^{1/2} \int_{\tau - \mu}^{\tau} (t - \tau)^{-1/2} \, d\tau < \left|m\right| \mu^{1/2}.
\]

Thus

\[
0 \leq v^*(t) \leq \left|m\right| + \mu^{1/2} \left|\lambda^{-1}\right| \left||q^*|| \left||v^*|| + ||f|| + 4\right|u_\infty\right| + 4\mu^{1/2} \left|g'|| + \mu^{1/2} \left|m\right| \left|\lambda^{-1}\right| \left(||q^*|| + ||v^*||\right) + F(m, \mu), \tag{2.9}
\]

where \( F \) is the maximum of \( \left|m\right| a(\mu)/b(\mu) \) and \( \left|m\right| \mu^{1/2} \). From (2.9) we get

\[
1 - \mu^{1/2} \left|\lambda^{-1}\right| \left(||q^*|| + ||f|| + 4\right|u_\infty\right| + 4\mu^{1/2} \left|g'|| + \mu^{1/2} \left|m\right| \left|\lambda^{-1}\right| \left||q^*|| + F(m, \mu). \tag{2.10}
\]

We select \( \mu \) so that the bracket on the left of (2.10) is positive. Then (2.10) implies that \( v^*(t) \) is bounded as \( t \to \sigma \) so that (2.6c) leads to a contradiction. Thus the only possibility is (2.6e).

We show now by an example that the third possibility occurs:

\[
u_{xx} = u, \quad 0 < x < s(t); \quad u(x, 0) = \varphi(x), \quad u_x(0, t) = 0, \tag{2.11}
\]

\[
u(s(t), t) = 0, \quad -\lambda s'(t) + u_x(s(t), t) = 0, \quad s(0) = a.
\]

Here \( \varphi(x) \geq 0, \varphi(a) = 0 \). We may suppose the solution reflected across the \( t \) axis. Since \( u(\pm s(t), t) = 0 \) and \( u(x, 0) = \varphi(x) \geq 0 \) we conclude from the maximum principle that \( u(x, t) \geq 0 \). Thus \( u_x(s(t), t) \leq 0 \) and therefore \( s'(t) \leq 0 \). When \( g(t) = 0 \) and \( \lambda(t) = \lambda \) constant we get, by an application of Green’s theorem to (1.1) [10, p. 55],

\[
s(t) = a + \lambda^{-1}H(t) + \lambda^{-1} \int_{0}^{s(t)} u(x, t) \, dx.
\]

For (2.11) we get then

\[
\lambda a - \int_{0}^{a} \varphi(x) \, dx = \lambda s(t) - \int_{0}^{s(t)} u(x, t) \, dx. \tag{2.12}
\]

Suppose the left side of (2.12) is negative; then we can eliminate cases (a) and (b) of the
theorem. In case (b), σ is finite and \( s(\sigma) = 0 \), we have only to let \( t \to \sigma \) to get a contradiction (\( u(x, t) \) is bounded by the maximum principle). Suppose we have case (a): a solution exists for all \( t \). We define \( u^0 \) to be the solution of

\[
\frac{\partial u}{\partial t} = u_x, \quad -a < x < a, \quad t > 0; \quad u(\pm x, 0) = \varphi(x), \quad u(\pm a, t) = 0.
\]

Then since \( u''(\pm s(t), t) > 0 \) and \( u^0 \) coincides with \( u \) at \( t = 0 \) we have, by the maximum principle, \( u^0 \geq u \geq 0 \). Thus

\[
0 \leq \int_0^a u(x, t) \, dx \leq \int_0^a u^0(x, t) \, dx < \int_0^a u^0(x, t) \, dx.
\]

(2.13)

The right side of (2.13) goes to 0 as \( t \to \infty \). Letting \( t \to \infty \) in (2.12) we get

\[
\lambda a - \int_0^a \varphi(x) \, dx = \lambda s(\infty),
\]

a contradiction since \( s(\infty) \geq 0 \). Thus case (c) of the theorem applies in this example. Since \( s'(t) \leq 0 \) it is clear that \( s(\sigma-) \) exists and is positive.

We have noted that case (c) of the theorem does not hold when conditions (1.3) are satisfied. We may also exclude possibility (c) in the case \( f(t) \geq 0, \varphi(x) \leq \varphi(a) \), and \( g'(t) \geq 0 \). Then \( u(x, t) \leq g(t) \); this follows from Lemma 1 of [12] applied to \( g(T) - u(x, t) \) for an arbitrary fixed \( T \) and for \( 0 \leq t \leq T \). Thus \( v(t) \geq 0 \) and we cannot have possibility (c). We may interpret \( 0 < x < s(t) \) as solid and \( x \geq s(t) \) as liquid, where the melting temperature \( g(t) \) is nondecreasing, heat is withdrawn at the rate \( f(t) \) at the fixed face, and there is flux \( q(t) \) at the moving interface.

References


