UNIFORM BOUNDEDNESS THEOREM FOR A NONLINEAR MATHEIU EQUATION*

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Bolotin [1, Ch. 3–Ch. 6] has derived the equation

$$f''(t) + 2\varepsilon f'(t) + \Omega^2(1 - 2\mu \cos \theta t)f(t) + \psi(f(t), f'(t), f''(t)) = 0$$

(1)

to describe the motion of a parametrically excited pin-ended elastic column. The function $f(t)$ is related to the displacement at time $t$ of the column from its undisturbed position as follows. Let $l$ be the length of the column and $x$ the displacement at height $h$. Then $x = f(t) \sin (\pi h/l)$. In a recent paper Genin and Maybee [2], using an energy function technique, have shown that all solutions are bounded when (1) is of the form

$$f'' = -(2\varepsilon f' + \Omega^2(1 - 2\mu \cos \theta t)f + \gamma f^3 + 2\varepsilon f'f'' + 2kf^2)/(1 + 2k^2)$$

(2)

and certain restrictions are placed on the parameters $\varepsilon, \Omega, \mu, \theta, \gamma, \varepsilon_1$, and $k$. In this note a theorem is presented which shows that every solution of (2) has a bound which is independent of the solution’s initial values. This result is obtained under less restrictive conditions than those used in [2].

THEOREM. There is a bounded region $A$ in the $xy$-plane such that if $f(t)$ satisfies (2), $f(t_0) = f_0$, and $f'(t_0) = f'_0$, then $(f(t), f'(t)) \in A$ for all $t \geq T$ where $T$ is a finite value which depends on $t_0, f_0$, and $f'_0$. The conditions placed on the parameters of (2) are:

$$\varepsilon, \gamma > 0; \varepsilon_1, \kappa \geq 0; \text{ and } \varepsilon_1 > 0 \text{ if } \kappa > 0.$$

Proof. Let

$$\Phi(x, y) = (x^2 + bxy + cy^2)(1 + 2\kappa x^2) + (b\varepsilon + c\Omega^2 - 1)x^2 + \frac{1}{2}(b\varepsilon_1 + c\gamma - 4\kappa)x^4$$

where $b$ and $c$ are chosen so that $b > 1/\varepsilon$ and $c > \max(b^2/4, b/(4\varepsilon), (4\kappa - b\varepsilon_1)/\gamma, \delta)$ where

$$\delta = 0 \quad \text{if } \kappa = 0,$$

$$= b\kappa/\varepsilon_1 \quad \text{if } \kappa > 0.$$ 

Since $4c > b^2$, $x^2 + bxy + cy^2$ is a positive definite form. Also $b\varepsilon + c\Omega^2 - 1 > 0$ and $b\varepsilon_1 + c\gamma - 4\kappa > 0$. Hence $\Phi$ is a positive definite function of $x$ and $y$. It is easily seen that, for any constant $C > 0$, the contour line $\Phi(x, y) = C$ is a single simple closed curve about the origin.

Let $f(t)$ satisfy (2). Then

$$\frac{d}{dt} [\Phi(f(t), f'(t)) = \frac{\partial \Phi}{\partial f} f' + \frac{\partial \Phi}{\partial f'} f'' = -(4\varepsilon \mu - b)f'^2 - 4(c\varepsilon_1 - b\kappa)f'^2 + b\Omega^2 f^2 - b\gamma f^4 + 2\Omega^2 \mu \cos \theta t(bf^2 + 2cf^2).$$

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Using $4\epsilon - b > 0$ we have
\[
\frac{d\Phi}{dt} = -\{4(4\epsilon - b)f'' + 4\Omega^2\mu c \cos (\theta t)f' + (4\Omega^4\mu^2c^2/(4\epsilon - b))f^2 \}
- \{4(c\epsilon - b\kappa)f'' + b\gamma f'^2 - [b\Phi^2(2\mu \cos \theta t - 1) + 4\Omega^4\mu^2c^2/(4\epsilon - b)]f^2 \}.
\]

Since $|\cos \theta t| \leq 1$,
\[
\frac{d\Phi}{dt} \leq -\{(4\epsilon - b)^{1/2} |f'| - (2\Omega^2\mu c/(4\epsilon - b)^{1/2}) |f| \}^2
- \{4(c\epsilon - b\kappa)f'' + b\gamma f'^2 - [b\Phi^2(2\mu - 1) + 4\Omega^4\mu^2c^2/(4\epsilon - b)]f^2 \}.
\] (3)

By examining the possibilities $c\epsilon - b\kappa = 0$ or $c\epsilon - b\kappa > 0$ and $\Omega^2\mu = 0$ or $\Omega^2\mu \neq 0$, it follows from (3) that $d\Phi/dt$ may be nonnegative only in a bounded region of the $xy$-plane. Let $A$ be the region bounded by a contour line of $\Phi$ which encloses in its interior all points at which $d\Phi/dt \geq 0$. Since $d\Phi/dt$ is negative and bounded away from 0 for points not in $A$, it follows that after some finite time, depending on $t_0$, $f_0$, and $f'_0$, the point $(f(t), f'(t))$ will enter and remain in $A$.

It may also be seen from (3) that the trivial solution has global asymptotic stability when $b\Omega^2(2\mu - 1) + 4\Omega^4\mu^2c^2/(4\epsilon - b) \leq 0$. This inequality is achieved when $|\mu|$ is sufficiently small.

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References