A BOUND ON THE ERROR IN REISSNER'S THEORY OF PLATES

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Introduction. In a recent paper [1] we derived an expression for the mean square error in stresses obtained from the solution of a boundary-value problem in the classical theory of plates with respect to the solution of a corresponding problem in the theory of elasticity. Further, the relative mean square error was bounded by a quantity proportional to the plate thickness. The derivation in [1] employs the hypersphere theorems of Prager and Synge [2], [3] in elasticity which lead to the equations of plate theory via energy minimization techniques.

Subsequently, Simmonds [4] bounded the relative mean square error by a quantity proportional to the plate thickness squared for isotropic plates. The derivation of this improved bound is based on a direct application of the main hypersphere theorem [2] with a specially constructed kinematically admissible stress field of a more elaborate form than in [1].

In the present note we derive an expression for the mean square error and a bound on the relative error for Reissner's theory of plates [5] which includes the effect of shear deformation. The derivation is similar to that of [1], although we employ a kinematically admissible stress field of nearly the same form as in [4]. The relative error is again bounded by a quantity proportional to plate thickness squared. This lack of essential improvement over the error bound for classical theory is not surprising in view of the well-known fact that Reissner's theory offers improvement over classical theory only near the edge of the plate.

Of particular interest is the value obtained here for the numerical constant in the constitutive equation for the shear stress resultant, namely \( \frac{1}{3} \). This value is in agreement with Reissner's original derivation [5] and most subsequent derivations, e.g., the direct derivation of Green [7]. In the present derivation the value \( \frac{1}{3} \) follows from minimization of both the potential energy and the complementary energy whose sum is the mean square error in stress [1], [3]. Therefore, \( \frac{1}{3} \) can be regarded as the best value for the shear constant within the context of mean square error minimization for static problems. Also, we find that the constitutive equation for stress couple need not include a term proportional to the normal surface load, as in [5] and [7]. Finally, the error bound of Simmonds [4] for classical theory of isotropic plates is extended to anisotropic plates with midsurface elastic symmetry.

For brevity we do not repeat Secs. 1 and 2 of [1] which deal, respectively, with function space concepts in elasticity and the statement of a class of boundary value problems.

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1 An exception is the value \( r^*/12 \) proposed by Mindlin [6] for dynamic problems. The present results are restricted to statics by the nature of the function space concepts in [2], [3].
problems for plates in elasticity. Instead reference will be made to results from [1] as needed in what follows.

**Potential energy.** The equations of Reissner’s theory will be derived here by minimization of potential energy using a kinematically admissible stress field.

Guided by solutions to plate problems in elasticity [8], we begin with a displacement field of the form

\[ u_a = \beta_a x_3 + f_a (x_3^2 - \frac{3}{2} h^2 x_3), \]
\[ u_3 = w + g(x_3^2 - \frac{3}{2} h^2), \]

where \( \beta_a, f_a, w \) and \( g \) are independent of the coordinate \( x_3 \) normal to the middle surface. The \( x_3 \) coefficients in (1) are chosen in view of the relations

\[ \int_{-h}^{h} x_3^2 (x_3^2 - \frac{3}{2} h^2 x_3) \, dx_3 = \int_{-h}^{h} (1 - x_3^2/h^2)(x_3^2 - \frac{3}{2} h^2) \, dx_3 = 0. \]

Thus, \( \beta_a \) and \( w \) may be interpreted as resultant displacements since

\[ \frac{3}{2h^3} \int_{-h}^{h} u_a x_3 \, dx_3 = \beta_a, \quad \frac{3}{4h} \int_{-h}^{h} (1 - x_3^2/h^2) u_3 \, dx_3 = w. \]

In order to be kinematically admissible, the displacement field (1) must meet the boundary conditions of the elasticity problem on the portion of the boundary \( S_u \). For this to be possible, a slight modification of the displacement boundary conditions in [1] is required. Specifically, (2.4) and (2.5) of [1] must be replaced by

\[ u_a = \beta_a^* x_3 + f_a^*(x_3^2 - \frac{3}{2} h^2 x_3), \quad u_3^* = w^* + g^*(x_3^2 - \frac{3}{2} h^2). \]

Then (1) satisfies the boundary conditions on \( S_u \) provided that

\[ \beta_a = \beta_a^*, \quad w = w^* \text{ on } C_u, \]
\[ f_a = f_a^*, \quad g = g^* \text{ on } C_u. \]

Proceeding from (1) as in [1] we obtain the following expression for the potential energy:

\[ V = V^{(o)} + \int \left( \frac{4h^7}{175} B_{a\beta\gamma} f_{a\beta\gamma} - \frac{3}{2} h^2 pg \right) \, dx_1 \, dx_2, \]

where

\[ V^{(o)} = \int \left\{ \frac{3}{2} h^3 B_{a\beta\gamma} f_{a\beta\gamma} + \frac{3}{2} h^3 B_{a\beta3} \beta_{a\beta} g \right. \]
\[ + \frac{3}{2} h^3 B_{333} \beta_{a\beta} g^2 + B_{a333} [h(\beta_{a\beta} + w_{a\beta})(\beta_{a\beta} + w_{a\beta}) + f_3 h(\beta_{a\beta} + w_{a\beta})(3f_{a\beta} + g_{a\beta}) \]
\[ + f_3 h(3f_{a\beta} + g_{a\beta})(3f_{a\beta} + g_{a\beta})] - pw \right\} \, dx_1 \, dx_2 - \int_{c_u} \left( \beta_a M_a^* + wQ^* \right) \, ds. \]

On minimization of \( V^{(o)} \) by the calculus of variations, the Euler equations can be written

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2 This form for the displacement field differs only slightly from that of Simmonds [4]. The essential difference from the displacement field of [1] is the \( f_a \) term. Here Greek indices take values 1 and 2 and repeated indices imply summation.

3 These same resultants or weighted displacements appear in the direct derivation of Reissner’s theory given by Green [7].
as
\[ g = \frac{B_{\beta\xi\eta}}{2B_{3333}} \beta_{\alpha,\xi}, \quad f_{\alpha} = -\frac{1}{2}g_{,\xi} - \frac{5}{12h^2} (\beta_{\alpha} + w_{,\alpha}) \]  
(8)
and
\[ M_{\alpha\beta,\xi} = Q_{\alpha} = 0, \quad -Q_{,\alpha,\xi} = p, \]  
(9)
where
\[ M_{\alpha\beta} = \frac{t^3}{12} \beta_{\alpha\beta,\xi} \gamma, \quad Q_{\alpha} = \frac{5}{6}tB_{\alpha\beta\gamma}(\beta_{\alpha} + w_{,\beta}), \]
\[ t = 2h, \quad B_{\alpha\beta\gamma} = B_{\alpha\beta\gamma1} - B_{\alpha\beta33}B_{\alpha\gamma33}/B_{3333}. \]  
(10)
Here \( t \) is the thickness of the plate. Further, the natural boundary conditions are
\[ M_{\alpha\beta,\xi} = M^*_\alpha, \quad Q_{,\alpha,\xi} = Q^* \text{ on } C_\gamma. \]  
(11)
Eqs. (9), (10), (5) and (11) are the equilibrium equations, constitutive relations and boundary conditions of Reissner’s theory of plates [5]. The functions \( f^*_\alpha \) and \( g^* \) in the displacement boundary conditions (4) of the elasticity problem are determined by (6) and (8) after solution of the plate problem. From a practical viewpoint this is a minor defect since resultant displacements \( \beta^*_\alpha \) and \( w^* \) are specified on \( C_u \).

The kinematically admissible stresses corresponding to (1) can be written as
\[ \sigma'_{\alpha\beta} = \beta_{\alpha\beta,\xi} \beta_{\gamma,\xi,\tau} + B_{\alpha\beta,\xi} \beta_{\gamma,\xi,\tau}(x_3^3 - \frac{3}{2}h^2 x_3), \]
\[ \sigma'_{33} = B_{3333} \beta_{\gamma,\xi,\tau}(x_3^3 - \frac{3}{2}h^2 x_3), \]
\[ \sigma'_{\alpha,\beta} = \frac{4}{3}B_{\alpha333}(\beta_{\alpha} + w_{,\beta})(1 - x_3^2/h^2), \]  
(12)
where, by (8), (9) and (10),
\[ f_{\alpha,\beta} = \frac{B_{3333}}{6B_{3333}} \beta_{\gamma,\xi,\tau} - \frac{2}{3}A_{\alpha333} \beta_{\eta,\xi,\gamma,\tau} \]
Complementary energy. The equations of Reissner’s theory are derived here by minimization of complementary energy using a statically admissible stress field. The approach is similar to that of Reissner [5].

Guided by solutions to plate problems in elasticity [8], we begin with a stress field of the usual form
\[ \sigma'_{\alpha\beta} = \frac{3x_3^2}{2h^3} M_{\alpha\beta}, \quad \sigma'_{33} = \frac{3}{4h} \left( 1 - \frac{x_3^2}{h^2} \right) Q_{\alpha}, \]
\[ \sigma'_{\alpha3} = \frac{3}{4h} \left( 1 - \frac{x_3^2}{3h^2} \right) x_3 p + q, \]  
(13)
where \( p \) and \( q \) are related to the surface stresses as in [1] by
\[ p = \sigma^+_3 + \sigma^-_3, \quad q = \frac{1}{2}(\sigma^+_3 - \sigma^-_3). \]
This stress field is statically admissible; i.e., the equilibrium equations and the stress boundary conditions of the elasticity problem are satisfied provided that \( M_{\alpha\beta} \) and \( Q_{\alpha} \) satisfy (9) and (11). By (13), (4) and (1.18) of [1] the complementary energy can be written as
\[
V_c = V^{(0)}_c + \int \left[ \frac{3}{5h} A_{a333} M_{\beta} \eta + A_{3333} \left( \frac{17}{140} p^2 + q^2 \right) \right] dx_1 \, dx_2 ,
\]

where
\[
V^{(0)}_c = \int \left[ \frac{3}{4h^3} A_{a333} M_{\beta} M_{\gamma} \eta + \frac{6}{5h} A_{3333} Q_{\alpha} Q_{\beta} \right] dx_1 \, dx_2 - \int_{s_b} \left[ \beta^* M_{\alpha} n_{\beta} + w^* Q_{\alpha} n_{\beta} \right] ds.
\]

On minimization of \( V^{(0)}_c \) with (9) enforced by Lagrange multipliers \( \beta \) and \( w \), we obtain the constitutive equations (10) as Euler equations and the natural boundary conditions (5). Thus, we again have the equations of Reissner's theory of plates.

**Approximate stress field and error.** According to the hypersphere theorem [2], [3], if \( \vartheta' \) and \( \vartheta'' \) are statically and kinematically admissible stress fields, respectively, for a boundary value problem in elasticity with exact stress field \( \vartheta \), then the approximate stress field
\[
\vartheta_\Lambda = \frac{1}{2} (\vartheta' + \vartheta'')
\]
has a mean square error \( E \); i.e.,
\[
|| \vartheta_\Lambda - \vartheta || = E
\]
where \( E \) is given by
\[
E = || \frac{1}{2} (\vartheta' - \vartheta'') ||
\]
and further
\[
E = V_c + V_p.
\]

Here \( || \vartheta || \) denotes the mean square norm defined in [1], [2] and [3]. By the bound (1.16) of [1], the relative error is of the form
\[
E/|| \vartheta || = n + \Theta(n^2), \quad n = E/(\vartheta' \cdot \vartheta'')^{1/2},
\]
where the inner product \( (\vartheta' \cdot \vartheta'') \) is defined in [1], [2] and [3]. In the present case, the error in the approximate stress field given by (12), (13) and (15) is
\[
E^2 = \int \left[ \frac{4h^2}{175} B_{a\beta\gamma} f_{a\beta\gamma} + h A_{3333} \left( \frac{17}{140} p^2 + q^2 \right) \right] dx_1 \, dx_2 ,
\]
which can be deduced from (17) using (12), (13) and (1.4) of [1], or from (18) using (7), (14) and the easily verified relations
\[
V_p^{(0)} + V_c^{(0)} = 0, \quad 3 A_{a333} M_{\alpha} = 4h^3 g.
\]

Further, by (12), (13) and (1.9) of [1]
\[
\vartheta' \cdot \vartheta'' = \int \left[ \frac{3}{4h^3} A_{a333} M_{\alpha} M_{\gamma} \eta + \frac{3}{10h} A_{3333} M_{\alpha} \eta + \frac{6}{5h} A_{3333} Q_{\alpha} Q_{\beta} \right] dx_1 \, dx_2 .
\]

Thus, by (9), (10), (19), (21) and (22), we obtain the following estimate for the relative error:
\[
E/|| \vartheta || = Ch^2 + \Theta(h^4),
\]

\footnote{In view of the second of (21), the \( p \) terms in \( V_p + V_c \) cancel whether or not they are included in \( V_p^{(0)} \) and \( V_c^{(0)} \). Therefore, it is not necessary to include these \( p \) terms in \( V_p^{(0)} \) and \( V_c^{(0)} \) which would lead to a \( p \) term in the constitutive equation for \( M_{\alpha} \) as in [5], [7].}
where
\[
C^a = \int \left[ \frac{4}{175} B_{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta} + \left( \frac{17}{140} + \gamma^2 \right) \frac{4}{9} A_{3333} \right. \\
\left. \cdot \left( \tilde{B}_{\alpha\beta\gamma\delta} \beta_{\alpha\beta\gamma\delta} \right)^2 \right] \, dx_1 \, dx_2 / \int \frac{1}{3} \tilde{B}_{\alpha\beta\gamma\delta} \beta_{\alpha\beta\gamma\delta} \, dx_1 \, dx_2 , \quad \gamma = q/p.
\]

Further, an explicit bound on the relative error follows immediately from (20), (22) and (1.16) of [1].

**Classical plate theory.** We indicate how the foregoing derivation can be modified to obtain results for classical plate theory. The derivation from complementary energy is the same as in [1] and need not be repeated here. In the derivation from potential energy using the displacement field (1), \( \beta_\alpha \) must be related to \( w \) before the potential energy is minimized, for otherwise Reissner’s theory would result as in the foregoing. As suggested by Simmonds [4], a suitable relation between \( \beta_\alpha \) and \( w \) follows from setting
\[
\sigma''_{\alpha\beta} = \sigma'_{\alpha\beta} = \frac{3}{4h} \left( 1 - x_3^2/h^2 \right) Q_\alpha ;
\]
i.e.,
\[
\beta_\alpha = -w_\alpha + \frac{12}{5h} A_{3333} Q_\beta ,
\]
\[
\gamma_{\alpha} = -\frac{1}{3} B_{\alpha\beta\gamma\delta} - \frac{1}{h^3} A_{3333} Q_\beta .
\]

Further, in order to make \( \sigma''_{\alpha\beta} \) small we set
\[
g = B_{\alpha\beta\gamma\delta} w_{\gamma\delta}/2B_{3333} .
\]

Then the equations of the classical theory of plates follow from minimization of the same \( V^{\text{pot}} \) as in [1]. In particular, the constitutive equations read
\[
M_{\alpha\beta} = -\frac{t^3}{12} B_{\alpha\beta\gamma\delta} w_{\gamma\delta} , \quad Q_\alpha = -\frac{t^3}{12} B_{\alpha\beta\gamma\delta} w_{\gamma\delta} .
\]

The kinematically admissible stress field is
\[
\sigma'''_{\alpha\beta} = \sigma'_{\alpha\beta} + B_{\alpha\beta\gamma\delta} \left[ \frac{4}{5} A_{3333} Q_\gamma \, x_3 + f_{\gamma\delta}(x_3^2 - \frac{2}{3} h^2 x_3) \right],
\]
\[
\sigma'''_{\alpha3} = \sigma'_{\alpha3} , \quad \sigma'''_{33} = B_{3333} \left[ \frac{4}{5} A_{3333} Q_\gamma \, x_3 + f_{\gamma\delta}(x_3^2 - \frac{2}{3} h^2 x_3) \right].
\]

By (17), (25), (27) and (28), the mean square error in the approximate stress field given by (13), (15) and (28) is
\[
E^2 = h^7 \int \left[ \frac{64}{75} B_{\alpha\beta\gamma\delta} \psi_{\alpha\beta\gamma\delta} + \frac{4}{175} B_{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta} \right. \\
\left. + A_{3333} \left( \frac{17}{140} + \gamma^2 \right) \frac{4}{9} (B_{\alpha\beta\gamma\delta} w_{\alpha\beta\gamma\delta})^2 \right] \, dx_1 \, dx_2 ,
\]

\footnote{This equation can also be obtained as an Euler equation on minimization of part of the potential energy.}
where

\[ \psi_a = A_{\alpha \beta \gamma} \bar{B}_{\alpha \beta \gamma} w_{\alpha \beta \gamma} \]

Further

\[ d' \cdot d'' = \frac{h^3}{3} \int \bar{B}_{\alpha \beta \gamma} w_{\alpha \beta \gamma} \, dx_1 \, dx_2 \]

and thus an estimate for relative mean square error of the form (23) follows from (19), (29) and (30). The foregoing results for classical theory of plates agree in essence with the results of Simmonds [4] for isotropic plates.

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References