TEMPERATURE OF A NONLINEARLY RADIATING SEMI-INFINITE SOLID*

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1. Introduction. Let $T(x, t)$ be the temperature of a semi-infinite heat-conducting solid occupying the half-space $x \geq 0$. We suppose that its surface radiates energy at a rate proportional to $[T(0, r)]^n$ and that the surface is heated by a source at a rate proportional to a given function $f(t)$. Here $n$ is a positive constant, the value $n = 1$ corresponding to Newton's law of cooling and $n = 4$ to Stefan's radiation law. If $T = 0$ initially, then for $t > 0$, $T$ is determined by the following initial boundary value problem:

$$T_t(x, t) = T_{xx}(x, t), \quad x > 0, \quad t > 0, \quad (1.1)$$
$$T_x(0, t) = \alpha T(0, t) - f(t), \quad t > 0, \quad (1.2)$$
$$T(x, 0) = 0, \quad x > 0, \quad (1.3)$$
$$T \to 0 \text{ as } x \to \infty, \quad t \geq 0. \quad (1.4)$$

Here $\alpha > 0$ is a given constant.

This problem has been considered by Mann and Wolf [1], Roberts and Mann [2] and Padmavally [3], while Friedman [4] has considered more general problems of a similar kind. From their work we can conclude that if $f(t)$ is a piecewise continuous bounded function then the above problem has a solution and it is unique. In addition Padmavally [3] has shown that if $f(t)$ is nondecreasing in the interval $0 < t < \tau$ then $T(0, t)$ is also nondecreasing in this interval.

Our aim is to obtain more detailed information about the surface temperature $T(0, t)$ when $f(t) > 0$ and $f(t)$ is integrable. First we shall obtain a sequence of upper and lower bounds on $T(x, t)$, which incidentally provide a constructive proof of its existence, and we shall also show its uniqueness. Then we shall show that as $t \to \infty$, $T(0, t) \sim \pi^{1/2} E(\infty)t^{-1/2}$ where $E(\infty)$ is the net energy flux into the solid through the surface. Furthermore, we shall show that $E(\infty) > 0$ for $n \geq 3$ while $E(\infty) = 0$ for $n \leq 2$. Thus for $n \geq 3$ some of the energy which enters the solid remains there, while for $n \leq 2$ it is all ultimately radiated away. We shall also examine the behavior of $T(0, t)$ for small values of $t$ as well as for large and small values of $\alpha$.

2. Equivalent integral equation. A solution $T(x, t)$ of (1.1)--(1.4) can be represented in terms of $T(0, t)$ by the formula

$$T(x, t) = \int_0^t f(s)G_s(x, t, s) \, ds$$

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This formula is obtained by applying Green's theorem to $T(x, t)$ and the Green's function $G_\rho$ defined by the following linear problem:

$$G_\rho = G_\rho(x, t, s), \quad x > 0, \quad t > s > 0,$$  \hfill (2.2)

$$G_\rho(0, t, s) = \rho(t)G_\rho(0, t, s) - \delta(t - s), \quad t \geq s, \quad (2.3)$$

$$G_\rho(x, t, s) = 0, \quad t < s, \quad x \geq 0, \quad (2.4)$$

$$G_\rho(x, t, s) \to 0 \text{ as } x \to \infty, \quad t \geq s. \quad (2.5)$$

The nonnegative function $\rho(t)$ in (2.3) is arbitrary, and can be chosen to facilitate the analysis. Any solution $T(x, t)$ of (2.1) satisfies (1.1)—(1.4).

We now set $x = 0$ in (2.1) to obtain a nonlinear integral equation for $T(0, t)$:

$$T(0, t) = \int_0^t \rho(s)G_\rho(0, t, s) \, ds + \int_0^t \rho(s)G_\rho(x, t, s) \, ds, \quad t \geq 0. \quad (2.6)$$

Once $T(0, t)$ is found from (2.6), it can be used in (2.1) to yield a solution $T(x, t)$ of (1.1)—(1.4). Thus the problem is reduced to solving (2.6).

Let us denote by $u_0(x, t)$ the first term on the right side of (2.1), i.e.

$$u_0(x, t) = \int_0^t \rho(s)G_\rho(x, t, s) \, ds. \quad (2.7)$$

It is evident that $u_0$ is the solution of the linear problem (2.2)—(2.5) with $\delta(t - s)$ replaced by $\rho(t)$. Now (2.6) can be written in the form

$$T(0, t) = u_0(0, t) + \int_0^t \rho(s)G_\rho(x, t, s) \, ds. \quad (2.8)$$

When $\rho(t) = 0$, (2.6) and (2.8) become the following simple-looking equation:

$$T(0, t) = \pi^{-1/2} \int_0^t \rho(s)G_\rho(0, t, s)\, ds. \quad (2.9)$$

3. **Bounds on** $T(x, t)$. Let us define the sequences of functions $u_j$ and $\rho_j$ as follows:

$$u_j(x, t) = u_{j-1}(x, t), \quad j = 1, 2, \cdots, \quad (3.1)$$

$$\rho_0(t) = 0, \quad \rho_j(t) = \alpha[u_{j-1}(0, t)]^{n-1}, \quad j = 1, 2, \cdots. \quad (3.2)$$

By the maximum principle, $G_\rho \geq 0$ and then from (2.7) and the assumption that $f \geq 0$ we have $u_j \geq 0$. Now for any two functions $\rho(t)$ and $\bar{\rho}(t)$, the functions $u_0$ and $u_1$ given by (2.7) are related by the integral equation

$$u_1(x, t) = u_0(x, t) + \int_0^t [\rho(s) - \bar{\rho}(s)]u_0(0, s)G_\rho(x, t, s) \, ds. \quad (3.2)$$

From (3.2) it follows first that $u_1 \leq u_0$, and then that $u_1 \leq u_2 \leq u_0$. By induction we
0 \leq u_1 \leq u_3 \leq \cdots \leq u_{2i-1} \leq \cdots \leq u_2 \leq u_0, \quad x \geq 0, \quad t \geq 0. \quad (3.3)

The functions \( u_{2i-1} \) form a monotone increasing sequence bounded above by \( u_0 \), while the \( u_{2i} \) form a monotone decreasing sequence bounded below by zero. Thus both sequences converge to limits, \( u^0 \) and \( u^* \), defined by

\[
\lim_{i \to \infty} u_{2i-1} = u^0, \quad \lim_{i \to \infty} u_{2i} = u^*. \quad (3.4)
\]

By using (3.2) in a suitable way, we can show that \( u^* = u^0 = u(x, t) \), say, and that \( u(0, t) \) is the unique solution of (2.9). Furthermore, \( u(x, t) \) is the unique solution of (1.1)–(1.4). (See Appendix A for details.) Thus the sequence \( u_i(0, t) \) converges to the unique solution \( T(0, t) \) of (2.9), providing a constructive proof of its existence, as was shown by Mann and Wolf [1] for a different sequence. From (3.4) and (3.3) it follows that the \( u_{2i-1} \) form an increasing sequence of lower bounds on \( T(x, t) \) while the \( u_{2i} \) form a decreasing sequence of upper bounds:

\[
0 \leq u_1 \leq u_3 \leq \cdots \leq u_{2i-1} \leq \cdots \leq T \leq \cdots \leq u_2 \leq u_0, \quad x \geq 0, \quad t \geq 0, \quad (3.5)
\]

In particular, (3.5) yields \( T(x, t) \geq 0 \).

Another interesting lower bound on \( T(0, t) \) can be obtained by choosing \( \rho(t) = \rho^*(t) \) in (2.3) where

\[
\rho^*(t) = \alpha Mt^{-1}, \quad t > 0, \quad M > 0. \quad (3.6)
\]

In Appendix B we show that as \( t \to \infty \),

\[
u_*(0, t) \sim C_0 t^{-1/2}, \quad C^* > 0. \quad (3.7)
\]

We now use \( \rho^* \) and \( u_\rho \) in (2.8) to obtain

\[
T(0, t) = u_\rho(0, t) + \alpha \int_0^t \left( M s^{-1} - [T(0, s)]^{-1} \right) T(0, s) G_\rho(0, t, s) ds, \quad t \geq 0. \quad (3.8)
\]

Now \( T(0, t) \) is positive, bounded, and decays at least as fast as \( t^{-1/2} \) as \( t \to \infty \), as we see from (3.5) and (4.8). Therefore it is possible to choose \( M \) so large that \( Mt^{-1} - [T(0, t)]^{-1} \geq 0 \) for all \( t > 0 \) provided that \( n \geq 3 \). Then it follows from (3.8) and (3.7) that

\[
T(0, t) \geq u_\rho(0, t) \sim C_0 t^{-1/2}, \quad C^* > 0, \quad n \geq 3. \quad (3.9)
\]

We now assume that \( 0 \leq f(t) \leq C \) where \( C > 0 \). Then we define \( \mu \) and \( K \) by

\[
\mu = \alpha n K^{n-1}, \quad K = (C/\alpha)^{1/n}. \quad (3.10)
\]

Upon setting \( \rho = \mu \) in (2.8), we obtain

\[
T(0, t) = u_\mu(0, t) + \alpha(n - 1) K^n \int_0^t G_\mu(0, t, s) ds \quad (3.11)
\]

\[\]
\begin{align*}
T(0, t) & \leq u_\star(0, t) + \alpha(n - 1)K^n \int_0^t G_\star(0, t, s) \, ds \\
& \leq [C + \alpha(n - 1)K^n] \int_0^t G_\star(0, t, s) \, ds, \quad n \geq 1. \quad (3.12)
\end{align*}

In Appendix C we show that the integral in (3.12) is bounded above by \( \mu^{-1} \), so (3.12) becomes

\begin{align*}
T(0, t) & \leq K = (C/\alpha)^{1/n}, \quad n \geq 1. \quad (3.13)
\end{align*}

To obtain another lower bound we define \( \gamma \) by

\begin{align*}
\gamma = \frac{\alpha}{C^{1/n}} = \frac{\alpha}{nC^{1/2}}. \quad (3.14)
\end{align*}

Then we set \( \rho = \gamma \) in (2.8) and then use (3.13) to obtain

\begin{align*}
T(0, t) = u_\gamma(0, t) \\
+ \alpha \int_0^t \{ K^{n-1} - [T(0, s)]^{n-1} \} T(0, s) G_\gamma(0, t, s) \, ds \geq u_\gamma(0, t), \quad n \geq 1. \quad (3.15)
\end{align*}

The lower bound \( u_\gamma \) in (3.15) is given by (2.7). For any constant \( \gamma > 0 \), \( G_\gamma \) is given by

\begin{align*}
G_\gamma(0, t, s) = \pi^{-1} (t - s)^{-1/2} \int_0^\infty \frac{\xi^{1/2} e^{-\xi}}{\xi + \gamma^2 (t - s)} \, d\xi, \quad t > s, \quad \gamma \geq 0. \quad (3.16)
\end{align*}

We now use (3.16) in (2.7) and evaluate \( u_\gamma \) for \( t \) large. Then (3.15) yields

\begin{align*}
T(0, t) \geq u_\gamma(0, t) \sim C_\gamma t^{-3/2}, \quad C_\gamma > 0, \quad n \geq 1. \quad (3.17)
\end{align*}

4. Behavior of \( T(0, t) \) for \( t \to \infty \). By integrating (1.1) with respect to \( x \) from 0 to \( \infty \) and with respect to \( t \) from 0 to \( t \) and using (1.2)--(1.4), we obtain

\begin{align*}
\int_0^t \{ f(s) - \alpha T^n(0, s) \} \, ds = \int_0^\infty T(x, t) \, dx. \quad (4.1)
\end{align*}

The left side of (4.1) is \( E(t) \), the net energy flow into the solid up to time \( t \), while the right side is the energy in the solid at time \( t \). We have shown above that if \( f \geq 0 \) then \( T(x, t) \geq 0 \), and thus the right side of (4.1) is nonnegative. Therefore (4.1) yields

\begin{align*}
E(t) = \int_0^t \{ f(s) - \alpha T^n(0, s) \} \, ds \geq 0 \quad \text{if} \quad f \geq 0. \quad (4.2)
\end{align*}

From (4.2) we obtain

\begin{align*}
\int_0^\infty T^n(0, s) \, ds < \infty \quad \text{if} \quad \int_0^\infty f(s) \, ds < \infty. \quad (4.3)
\end{align*}

We can now determine the behavior of \( T(0, t) \) for \( t \to \infty \) by utilizing (4.3) to evaluate the integral in (2.9) asymptotically. We see at once that

\begin{align*}
T(0, t) \sim \pi^{-1/2} \int_0^\infty \{ f(s) - \alpha T^n(0, s) \} \, ds t^{-1/2} \sim \pi^{-1/2} E(\infty) t^{-1/2}. \quad (4.4)
\end{align*}

Upon using (4.4) in (3.9) we obtain

\begin{align*}
E(\infty) \geq \pi^{1/2} C^* > 0, \quad n \geq 3. \quad (4.5)
\end{align*}
By using (4.4) in (4.3), we see that when $E(\infty) > 0$ the integral of $T^n$ is finite only if $n > 2$. It follows that

$$E(\infty) = 0, \quad n \leq 2. \quad (4.6)$$

Thus (4.4) shows only that $T(0, t) = o(t^{-1/2})$ for $n \leq 2$. On the other hand, (3.17) shows that $T(0, t)$ does not decrease faster than $t^{-3/2}$ for $n \geq 1$.

When $n = 1$ the explicit solution of (2.8) is

$$T(0, t) = u_\alpha(0, t) \sim C_\alpha t^{-3/2}, \quad C_\alpha > 0, \quad n = 1, \quad \alpha > 0. \quad (4.7)$$

Thus for $n = 1$, $T(0, t)$ decays at the fastest rate permitted by (3.17). However if $\alpha = 0$, which we have hitherto excluded, then (2.9) shows that $T(0, t)$ is independent of $n$ and is given by

$$T(0, t) = u_0(0, t) \sim C_0 t^{-1/2}, \quad C_0 > 0, \quad \alpha = 0. \quad (4.8)$$

Comparison of (4.4) with (4.8) shows that for $n > 2$, $T(0, t)$ decays at the same slow rate $O(t^{-1/2})$ as if the boundary were not radiating. To understand this we write the radiation rate $\alpha T^n$ as $\alpha'(t)T$ with the effective radiation constant $\alpha'(t) = \alpha T^{n-1}$. Now for $n > 1$, $\alpha'(t)$ tends to zero as $t \to \infty$, so the boundary tends to behave as a nonradiating boundary ($\alpha = 0$) as $t \to \infty$. Evidently for $1 < n < 2$, $\alpha'(t)$ does not tend to zero fast enough to make $T(0, t)$ decay as slowly as $t^{-1/2}$, but for $n > 2$ it does.

5. Perturbation expansions. To find $T(0, t)$ for small values of $\alpha$, we use (2.9) and solve it by iterations. For $\alpha$ small we can write the results as

$$T(0, t) = u_0(0, t) - \alpha \pi^{-1/2} \int_0^t \frac{u_0(0, s)}{(t-s)^{1/2}} ds$$

$$+ \frac{n \alpha^2 - 1}{\pi} \int_0^t \frac{u_0^{-3/2}(0, s)}{(t-s)^{1/2}} \int_0^s \frac{u_0(0, r)}{(s-r)^{1/2}} dr ds + O(\alpha^2). \quad (5.1)$$

For $t$ small, we require $f(t)$ to be such that $u_0(0, t)$ has the expansion

$$u_0(0, t) = at^h + bt^g + O(t^h), \quad t \to 0, \quad g > h. \quad (5.2)$$

Then the iterative solution of (2.9) yields

$$T(0, t) = at^h + bt^g + O(t^h) - a \pi^{-1/2} \sum_{k=0}^n \frac{t^{k+1/2}}{t^{k+1/2}} \left[ 1 + O(t^{-h}) \right]$$

$$+ \alpha^2 \pi^{-1} a^{-1} I_n I_{2n-1} t^{(2n-1)h+1/2} \left[ 1 + O(t^{-h}) \right], \quad t \to 0. \quad (5.3)$$

Here we have introduced $I_\alpha$, defined by

$$I_\alpha = \int_0^1 \frac{s^\alpha}{(1-s)^{1/2}} ds. \quad (5.4)$$

To find $T(0, t)$ for $\alpha$ large, we first use the Abel inversion formula to solve (2.9) for $T^n$ in the form

$$T^\alpha(0, t) = \frac{t(t)}{\alpha} \frac{1}{\alpha \pi^{1/2}} \frac{d}{dt} \int_0^t (t-s)^{1/2} T(0, s) ds. \quad (5.5)$$
Then we iterate (5.5) to obtain

\[ T(0, t) = \alpha^{-1/n} [f(t)]^{1/n} \]

\[ \quad - \alpha^{-2/n} n^{-1/2} [f(t)]^{1/n-1} \frac{d}{dt} \int_0^t [f(s)]^{1/n} (t - s)^{-1/2} ds + O(\alpha^{-3/n}), \quad t > 0. \] (5.6)

The result (5.6) cannot be valid at \( t = 0 \) because \( f(0) \) may not be zero, whereas \( T(0, 0) \) must be zero. It is not valid for \( t \) large if \( f(t) \) decays too fast. Thus an initial layer expansion is required at and near \( t = 0 \), and another expansion may be needed for large \( t \), but we shall not determine it.

**Appendix A. Existence and uniqueness.** To show that \( u' \equiv u^0 \), we consider (3.2) with \( \rho(t) = \alpha[u_2(0, t)]^{-1} \) and \( \varphi(t) = \alpha[u_{2-1}(0, t)]^{-1} \). Then taking limits as \( j \rightarrow \infty \) yields the equation

\[ u'(x, t) - u^0(x, t) = \int_0^t [w'(0, s) - u^0(0, s)] \varphi(x, t, s) ds, \quad t \geq 0, \quad x \geq 0, \] (A.1)

where

\[ \varphi(x, t, s) = \frac{[u'(0, s)]^{-1} - [u^0(0, s)]^{-1}}{u'(0, s) - u^0(0, s)} u'(0, s) G_x(x, t, s) \geq 0. \] (A.2)

By setting \( x = 0 \) in (A.1) we obtain

\[ u'(0, t) - u^0(0, t) = \int_0^t [u'(0, s) - u^0(0, s)] \varphi(0, t, s) ds, \quad t \geq 0. \] (A.3)

This can be viewed as a homogeneous integral equation of the second kind for \( u'(0, t) - u^0(0, t) \) with \( \varphi(0, t, s) \) as the kernel. If we choose a \( t \) such that \( |u'(0, s) - u^0(0, s)| \leq |u'(0, t) - u^0(0, t)| \) \( \) for \( 0 \leq s \leq t \), then (A.3) yields

\[ |u'(0, t) - u^0(0, t)| \leq |u'(0, t) - u^0(0, t)| \int_0^t \varphi(0, t, s) ds. \] (A.4)

For \( t \) sufficiently small, say \( 0 \leq t \leq \varepsilon \), the integral in (A.4) is less than unity, which implies that \( u'(0, t) = u^0(0, t) \) for \( t \leq \varepsilon \). Using this fact in (A.3), we can show that \( u'(0, t) = u^0(0, t) \) in a larger interval. This procedure can be repeated to show that \( u'(0, t) = u^0(0, t) \) for all \( t \geq 0 \). Then (A.1) shows that \( u'(x, t) = u^0(x, t) \) for all \( x \geq 0, t \geq 0 \). Thus there is a common limit \( u(x, t) \), so

\[ u(x, t) = u'(x, t) = u^0(x, t), \quad x \geq 0, \quad t \geq 0. \] (A.5)

It follows from the definition (3.1) of \( u_i \) and from (3.2) that \( u \) and \( u_{i-1} \) satisfy

\[ u_i(x, t) = u_x(x, t) + \int_0^t \{ \rho(s) - \alpha[u_{i-1}(0, s)]^{-1} \} u_i(0, s) G_x(x, t, s) ds. \] (A.6)

Then since \( u_i \rightarrow u \) and \( u_{i-1} \rightarrow u \), it is clear from (A.6) that \( u \) satisfies (2.1).

To show that the nonnegative solution constructed above is unique, we assume that there are two solutions \( T_1 \) and \( T_2 \). By subtracting (2.9) for \( T_2 \) from (2.9) for \( T_1 \), we obtain
Now by the same arguments used above to show that \( u'(0, t) = u^0(0, t) \), it follows that \( T_1(0, t) = T_2(0, t) \). Then from (2.1) it follows that \( T_1(x, t) = T_2(x, t) \).

**Appendix B. Asymptotic behavior of \( u_\mu(0, t) \).** For establish the asymptotic property (3.7) for \( u_\mu(0, t) \), we consider the initial boundary value problem (2.2)-(2.5) for \( u_\mu \) with \( \rho(t) = \rho^*(t) = \alpha M t^{-1} \) and with \( \delta(t - s) \) replaced by \( f(t) \). Applying the Laplace transform to this problem yields

\[
\hat{u}_{\mu}(x, p) + p \hat{u}_{\mu}(x, p) = 0, \quad x > 0, \quad (B.1)
\]

\[
\hat{u}_{\mu}(0, p) = \alpha M \int_0^\infty e^{-pt} u_{\mu}(0, t) \, dt - f(p), \quad (B.2)
\]

\[
\hat{u}_{\mu}(x, p) \to 0, \quad x \to \infty. \quad (B.3)
\]

Here \( \hat{u}_{\mu}(x, p) \) and \( f(p) \) are defined by

\[
\hat{u}_{\mu}(x, p) = \int_0^\infty e^{-pt} u_{\mu}(x, t) \, dt, \quad f(p) = \int_0^\infty e^{-pt} f(t) \, dt. \quad (B.4)
\]

The solution of (B.1) satisfying (B.3) is

\[
\hat{u}_{\mu}(x, p) = A(p)e^{-p^{1/2}x}. \quad (B.5)
\]

Here \( A(p) = \hat{u}_{\mu}(0, p) \) must be determined from the boundary condition (B.2). Upon substitution of (B.5) into (B.2) we obtain

\[
-p^{1/2}A(p) = \alpha M \int_0^\infty e^{-pt} u_{\mu}(0, t) \, dt - f(p). \quad (B.6)
\]

Differentiation of (B.6) with respect to \( p \) yields

\[
-\frac{d}{dp} [p^{1/2}A(p)] = -\alpha M A(p) - \frac{d}{dp} f(p). \quad (B.7)
\]

The solution of (B.7) which satisfies (B.6) is

\[
A(p) = -p^{-1/2} \exp \left[ 2\alpha M p^{1/2} \right] \int_p^\infty \exp \left[ -2\alpha M \xi^{1/2} \right] \xi \, d\xi. \quad (B.8)
\]

As \( p \to 0 \), (B.8) implies that

\[
A(p) \sim p^{-1/2} \int_0^\infty \exp \left[ -2\alpha M \xi^{1/2} \right] \int_0^\infty t f(t)e^{-t\xi} \, dt \, d\xi \quad \text{as} \quad p \to 0. \quad (B.9)
\]

Then a classical asymptotic result on Laplace transforms shows that

\[
u_{\mu}(0, t) \sim C t^{-1/2} \quad \text{as} \quad t \to \infty, \quad C > 0. \quad (B.10)
\]

**Appendix C. Estimation of an integral.** To estimate the integral in (3.12) we consider (2.2)-(2.5) with \( \rho(t) = \mu = \text{constant} \). Upon integrating the differential equa-
tion (2.2) we obtain
\[ \int_0^{t^*} \int_0^\infty G_{\mu,t}(x, t, s) \, ds \, dx = \int_0^{t^*} \int_0^\infty G_{\mu,t}(x, t, s) \, ds \, dx = -\int_0^{t^*} G_{\mu,t}(0, t, s) \, ds \quad \text{(C.1)} \]
By virtue of the boundary condition (2.3) we then have
\[ \int_0^{t^*} \int_0^\infty G_{\mu,t}(x, t, s) \, ds \, dx = 1 - \mu \int_0^t G_{\mu,t}(0, t, s) \, ds. \quad \text{(C.2)} \]
Since \( G_{\mu}(x, t, s) \) depends on the difference \( t - s \), \( G_{\mu,t} = -G_{\mu,s} \) and (C.2) becomes
\[ 0 \leq \int_0^\infty G_{\mu}(x, t, 0) \, dx = 1 - \mu \int_0^t G_{\mu}(0, t, s) \, ds, \quad t > 0. \quad \text{(C.3)} \]
This gives the desired inequality
\[ \int_0^t G_{\mu}(0, t, s) \, ds \leq \mu^{-1}. \quad \text{(C.4)} \]

References