ON AN INTEGRAL EQUATION APPROACH TO
DISPLACEMENT PROBLEMS OF CLASSICAL ELASTICITY*

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1. Introduction. In a recent paper Kanwal [1] has established an integral equation
method for solving displacement problems of elasticity. The method is based on the
generalization of Green's procedure in potential theory. But it appears to have gone
unnoticed by Kanwal that his formulation has a direct bearing on Betti's method of
elasticity [2], which has long been known as a tensor counterpart of Green's procedure.

The purpose of the present note is to point this fact out and to show further that
the result obtained by Kanwal is identical with that of Betti. The note deals only with
the static case, but the analysis may be extended easily to the dynamic case.

2. Integral equation method. In the absence of body forces the Navier-Cauchy
equations in elastostatics are

\[(\lambda + \mu) \text{grad} \vartheta + \mu \nabla^2 \mathbf{u} = 0, \quad \vartheta = \text{div} \mathbf{u},\]

where \(\mathbf{u} (u_i; i = 1, 2, 3)\) is the displacement vector and \(\lambda, \mu\) are Lamé constants of the
material medium. The components \(u_i\) are functions of Cartesian coordinates \(x_i\). Let
the region under consideration be denoted by \(V\) and its bounding surface by \(S\). The
unit normal \(\mathbf{n} (n_i; i = 1, 2, 3)\) will be directed outward to \(S\). A brief description of
Kanwal's procedure in deriving the solutions of Eqs. (1) follows.

Choose a tensor function \(U (U_{ij}; i, j = 1, 2, 3)\) such that

\[U_{ij} = \frac{1}{8\pi\mu(\lambda + 2\mu)} \left\{ (\lambda + 3\mu) \frac{\delta_{ij}}{|\mathbf{P} - \mathbf{q}|} + (\lambda + \mu) \frac{(x_i - x_i^0)(x_j - x_j^0)}{|\mathbf{P} - \mathbf{q}|^3} \right\},\]

which corresponds to the \(i\)th component of the displacement at a point \(\mathbf{P}(x_i)\) by a unit
force applied in the \(j\)th direction at a point \(\mathbf{q}(x_j^0)\); \(\delta_{ij}\) is the Kronecker delta. From \(U_{ij}\)
we can compute dilatations \(\vartheta\), which are given by

\[\vartheta_i = \frac{\partial U_{ik}}{\partial x_k} = -\frac{1}{4\pi(\lambda + 2\mu)} \frac{x_i - x_i^0}{|\mathbf{P} - \mathbf{q}|^3}.\]

To find the displacement \(\mathbf{u}\) at an interior point \(\mathbf{P}\) we surround the point \(\mathbf{P}\) with an
infinitesimal sphere \(S_x\) (since \(\vartheta_i\) and \(U_{ij}\) are singular at \(\mathbf{P} = \mathbf{q}\) of radius \(\sigma\) so that the
sphere lies entirely in \(V\). Let the volume enclosed by the sphere be denoted by \(V_x\). Applying
Green's second identity to the functions \(u_i, U_{ij}\), we obtain

\[\int_{V - V_x} \{\mathbf{u} \cdot \nabla^2 U_{ij} - U_{ij} \nabla^2 \mathbf{u}_i\} \, dV = \int_{S + S_x} \left\{ u_i \frac{\partial U_{ij}}{\partial n} - U_{ij} \frac{\partial u_i}{\partial n} \right\} \, ds.\]
whence it can be derived that
\[
\int_{S+S_v} \left[ \mathbf{u} \cdot \left\{ \mu \frac{\partial \mathbf{U}}{\partial n} + (\lambda + \mu) \mathbf{n} \right\} - \mathbf{U} \cdot \left\{ \mu \frac{\partial \mathbf{u}}{\partial n} + (\lambda + \mu) \mathbf{n} \right\} \right] ds = 0. \tag{5}
\]
Evaluating the integral over \( S_v \) and letting \( \sigma \to 0 \), we may derive the displacement at any point \( P \) \((\subset V)\) as
\[
\mathbf{u}(P) = -\int_S \left[ \mathbf{u} \cdot \left\{ \mu \frac{\partial \mathbf{U}}{\partial n} + (\lambda + \mu) \mathbf{n} \right\} - \mathbf{U} \cdot \left\{ \mu \frac{\partial \mathbf{u}}{\partial n} + (\lambda + \mu) \mathbf{n} \right\} \right]. \tag{6}
\]
Now we show that Kanwal’s formulation (6) is identical with the classic result of Betti, which expresses the displacement at the point \( P \) as
\[
u_i(P) = \int_S \{ t_i U_{i,i} - u_i T_{i,i} \} \, ds, \tag{7}
\]
better known as Betti’s second identity [3]. The boundary tractions \( t (t_i ; i = 1, 2, 3) \) and \( T (T_{i,j} ; i, j, = 1, 2, 3) \) are computed from the displacements \( u \) and \( U \) respectively. Eq. (5) may be rewritten as
\[
\int_{S+S_v} \{ \mathbf{u} \cdot \mathbf{T} - t \cdot \mathbf{U} \} \, ds + \mu \int_{S+S_v} \{ \mathbf{u} \cdot (\mathbf{a} \wedge \mathbf{U}) - \mathbf{U} \cdot (\mathbf{a} \wedge \mathbf{u}) \} \, ds = 0, \tag{8}
\]
where \( \mathbf{a} = \nabla \wedge \mathbf{n} \). The second integral in Eq. (8) vanishes identically by Gauss’ divergence theorem, and thus Eq. (8) is identical with Eq. (2.3) of Rizzo [4] from which Betti’s second identity (7) follows immediately.

References