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AN EXTENSION OF LIÉNARD'S GRAPHICAL CONSTRUCTION*

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The free oscillations of a damped second-order mass-spring system are governed by an ordinary differential equation of the form

$$\ddot{x} + \phi(\dot{x}) + f(x) = 0, \tag{1}$$

where $-f$ is the spring force, $-\phi$ is the force due to damping and a dot denotes differentiation with respect to time. If we put $\dot{x} = v$, then (1) becomes

$$dv/dx = (-\phi(v) - f(x))/v, \tag{2}$$

and this latter equation is ordinarily interpreted as specifying a field of directions in the $x - v$ (phase) plane; the direction, dv/dx, at point $P = (x_0, v_0)$ is the slope of the solution curve through that point evaluated at P and is, from (2), equal to $(-\phi(v_0) - f(x_0))/v_0$.

The classical Liénard construction [1, pp. 31–36] deals only with the particular case where $f(x)$ is linear in x. Under this hypothesis and with proper scaling of the coordinates x and v, we may write (2) in the standard Liénard form:

$$dv/dx = (-\phi(v) - x)/v. \tag{3}$$

Our purpose in this note is to present a graphical construction which applies directly to the more general equation (2); that is, in the method to be given, $f(x)$ may be an arbitrary continuous nonlinear function. The method also enables one to treat the linear case $f(x) = \alpha x$ without the prior normalization ordinarily required to achieve (3). (See [2, pp. 438–440] for discussion of the appropriate scale changes.)

Extended method. We wish to detail a graphical method for constructing the value of dv/dx given in (2) without resorting to an elaborate plot of the isoclines. In Fig. 1, let $P = (x_0, v_0)$ denote a generic point at which the value of dv/dx is desired. First, two curves are plotted in the $x - v$ plane, $x = -\phi(v)$ which is the standard "Liénard characteristic" and $v = f(x)$ representing the spring characteristic. (The curve $v = f(x)$ shown in the figure is a typical "hard spring" nonlinearity.) In addition to these two curves, the line $v = x$ is also drawn in the $x - v$ plane.

For the construction, a horizontal line L is first drawn through P intersecting the curve $x = -\phi(v)$ in point R. The point R is then projected vertically on the x-axis to obtain point S with coordinates $(-\phi(v_0), 0)$. Note that the procedure coincides thus far with the usual Liénard construction.

Next, draw a vertical line through P intersecting the curve $v = f(x)$ in point Q. Then Q is projected horizontally to intersect the line $v = x$ in point Q'. Lastly Q' is projected vertically to intersect the horizontal line L through P in point P'. The required direction at (x_0, v_0) is then exhibited by a line perpendicular to the line $\overline{SP'}$.

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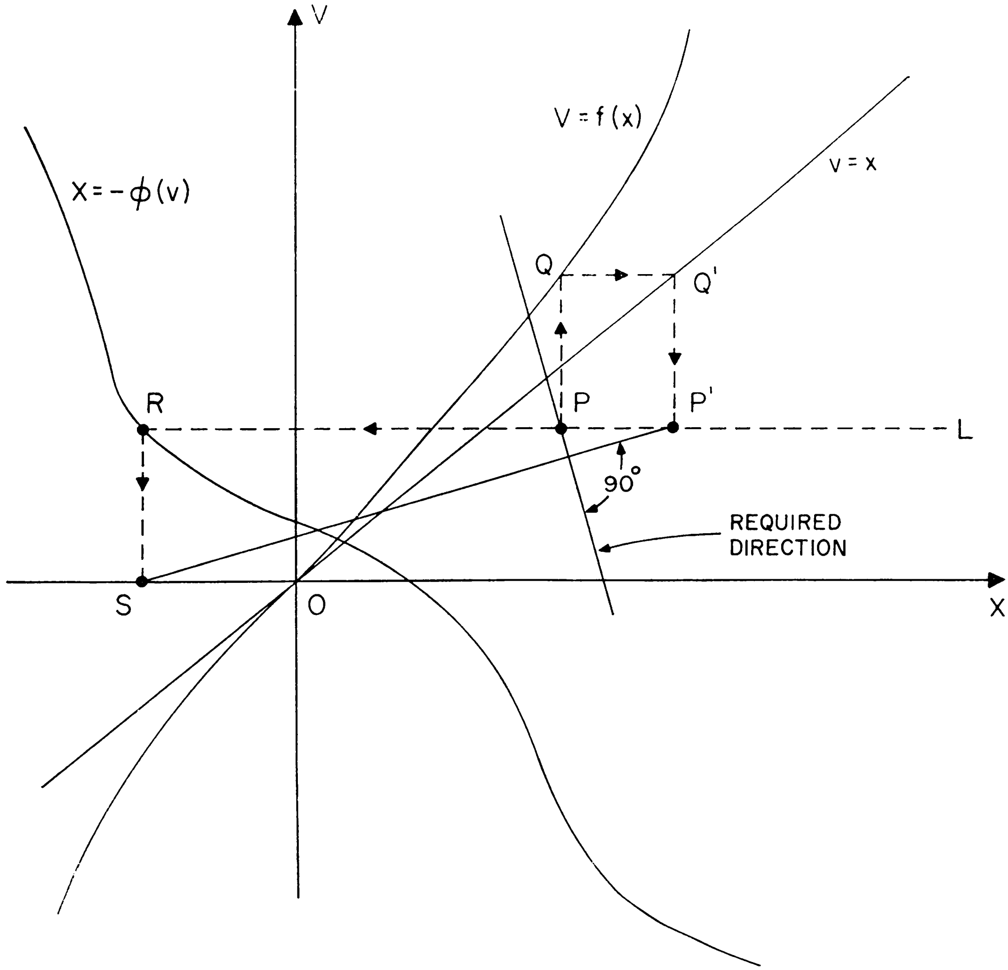

FIG. 1. Extended method.

The proof of this latter assertion is easily seen; for, point P' has coordinates $(f(x_0), v_0)$ by construction, while S has coordinates $(-\phi(v_0), 0)$. Thus, the line $\overline{SP'}$ has slope equal to $v_0/(f(x_0) + \phi(v_0))$, and hence a line perpendicular to $\overline{SP'}$ will have the slope $(-\phi(v_0) - f(x_0))/v_0$ as required by (2) for dv/dx at point $P = (x_0, v_0)$.

It is trivial to verify that the extended procedure reduces to the usual Liénard method when the specialization $f(x) = x$ is made.

The crux of the extended method lies in converting the ordinate $f(x_0)$ into an abscissa having the same value, essentially through reflection in the line $v = x$. This same idea may be used to treat the Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{4}$$

Here, the transformation, $y = x + F(x)$, where $F(x) = \int_0^x f(\xi)\, d\xi$, converts (4) into the pair of equations ([3], pp. 140–141)

$$dx/dt = y - F(x), \qquad dy/dt = -g(x), \tag{5}$$

or the single equivalent equation,

$$dx/dy = (F(x) - y)/g(x). \tag{6}$$

The appropriate construction is shown in Figure 2, where P with coordinates (y_0, x_0) is the point in the $y - x$ plane at which dx/dy is to be evaluated. As before, three curves are drawn in the $y - x$ plane, namely $y = g(x)$, $y = F(x)$ and the line $y = x$ needed to convert an abscissa value into a corresponding ordinate value. The steps in the construction are as follows:

(i) Project P horizontally to intersect the curve $y = F(x)$ in point R.
(ii) Determine S with coordinates $(F(x_0), 0)$ as the projection of R on the y-axis.
(iii) Project P horizontally to intersect the curve $y = g(x)$ at Q.
(iv) Project Q vertically to meet the line $y = x$ at Q' and then extend Q' horizontally to intersect a vertical line through P at P'.
(v) A line through P perpendicular to the line $\overline{SP'}$ gives the required direction (value of dx/dy) at point P.

The proof is obvious from the figure on noting that the slope of line $\overline{SP'}$ is

$$g(x_0)/(y_0 - F(x_0)).$$

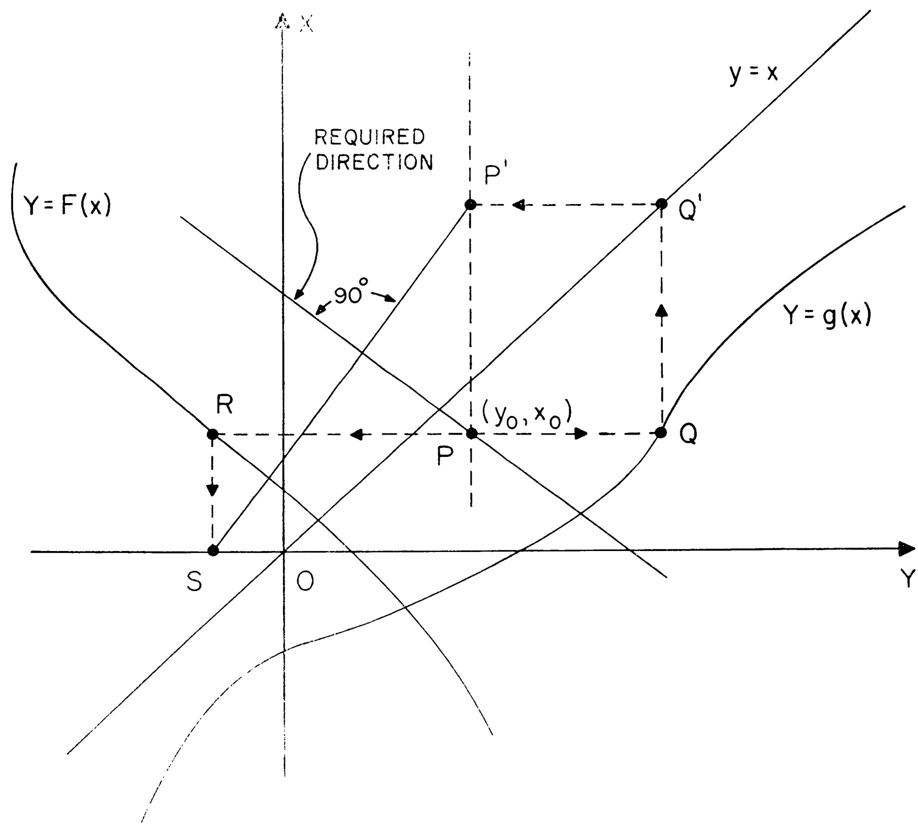

FIG. 2. Construction for Liénard equation.

More complicated versions of (5) and (6) can also be handled geometrically using the same ideas; however, the construction becomes correspondingly more complex, in general requiring one or more additional curves in the phase-plane diagram.

References

[1] J. J. Stoker, *Nonlinear vibrations in mechanical and electrical systems*, Interscience, New York, 1950
[2] T. E. Stern, *Theory of nonlinear networks and systems*, Addison–Wesley, Reading, Mass., 1965
[3] V. V. Nemyckiĭ and V. V. Stepanov, *Qualitative theory of differential equations*, GITTL, Moscow, 1949; English transl., Princeton Math. Series, no. 22, Princeton, N. J., 1960