AN EXTENSION OF LIÉNARD’S GRAPHICAL CONSTRUCTION*

By J. L. BROWN, JR. (Pennsylvania State University)

The free oscillations of a damped second-order mass-spring system are governed by an ordinary differential equation of the form

$$\ddot{x} + \phi(\dot{x}) + f(x) = 0,$$

(1)

where $-f$ is the spring force, $-\phi$ is the force due to damping and a dot denotes differentiation with respect to time. If we put $\dot{x} = v$, then (1) becomes

$$\frac{dv}{dx} = \frac{-(\phi(v) - f(x))}{v},$$

(2)

and this latter equation is ordinarily interpreted as specifying a field of directions in the $x - v$ (phase) plane; the direction, $dv/dx$, at point $P = (x_0, v_0)$ is the slope of the solution curve through that point evaluated at $P$ and is, from (2), equal to $-(\phi(v_0) - f(x_0))/v_0$.

The classical Liénard construction [1, pp. 31–36] deals only with the particular case where $f(x)$ is linear in $x$. Under this hypothesis and with proper scaling of the coordinates $x$ and $v$, we may write (2) in the standard Liénard form:

$$\frac{dv}{dx} = \frac{-(\phi(v) - x)}{v}.$$  

(3)

Our purpose in this note is to present a graphical construction which applies directly to the more general equation (2); that is, in the method to be given, $f(x)$ may be an arbitrary continuous nonlinear function. The method also enables one to treat the linear case $f(x) = \alpha x$ without the prior normalization ordinarily required to achieve (3). (See [2, pp. 438–440] for discussion of the appropriate scale changes.)

Extended method. We wish to detail a graphical method for constructing the value of $dv/dx$ given in (2) without resorting to an elaborate plot of the isoclines. In Fig. 1, let $P = (x_0, v_0)$ denote a generic point at which the value of $dv/dx$ is desired. First, two curves are plotted in the $x - v$ plane, $x = -\phi(v)$ which is the standard “Liénard characteristic” and $v = f(x)$ representing the spring characteristic. (The curve $v = f(x)$ shown in the figure is a typical “hard spring” nonlinearity.) In addition to these two curves, the line $v = x$ is also drawn in the $x - v$ plane.

For the construction, a horizontal line $L$ is first drawn through $P$ intersecting the curve $x = -\phi(v)$ in point $R$. The point $R$ is then projected vertically on the $x$-axis to obtain point $S$ with coordinates $(-\phi(v_0), 0)$. Note that the procedure coincides thus far with the usual Liénard construction.

Next, draw a vertical line through $P$ intersecting the curve $v = f(x)$ in point $Q$. Then $Q$ is projected horizontally to intersect the line $v = x$ in point $Q'$. Lastly $Q'$ is projected vertically to intersect the horizontal line $L$ through $P$ in point $P'$. The required direction at $(x_0, v_0)$ is then exhibited by a line perpendicular to the line $SP'$.

* Received February 13, 1971.
The proof of this latter assertion is easily seen; for, point $P'$ has coordinates $(f(x_0), v_0)$ by construction, while $S$ has coordinates $(-\phi(v_0), 0)$. Thus, the line $SP'$ has slope equal to $v_0/(f(x_0) + \phi(v_0))$, and hence a line perpendicular to $SP'$ will have the slope $(-\phi(v_0) - f(x_0))/v_0$ as required by (2) for $dv/dx$ at point $P = (x_0, v_0)$.

It is trivial to verify that the extended procedure reduces to the usual Liénard method when the specialization $f(x) = x$ is made.

The crux of the extended method lies in converting the ordinate $f(x_0)$ into an abscissa having the same value, essentially through reflection in the line $v = x$. This same idea may be used to treat the Liénard equation:

$$x + f(x)x + g(x) = 0. \quad (4)$$

Here, the transformation, $y = x + F(x)$, where $F(x) = \int_0^x f(\xi) \, d\xi$, converts (4) into the pair of equations ([3], pp. 140–141)

$$dx/dt = y - F(x), \quad dy/dt = -g(x), \quad (5)$$
or the single equivalent equation,

\[ \frac{dx}{dy} = \frac{(F(x) - y)}{g(x)}. \tag{6} \]

The appropriate construction is shown in Figure 2, where \( P \) with coordinates \((y_0, x_0)\) is the point in the \( y - x \) plane at which \( \frac{dx}{dy} \) is to be evaluated. As before, three curves are drawn in the \( y - x \) plane, namely \( y = g(x) \), \( y = F(x) \) and the line \( y = x \) needed to convert an abscissa value into a corresponding ordinate value. The steps in the construction are as follows:

(i) Project \( P \) horizontally to intersect the curve \( y = F(x) \) in point \( R \).
(ii) Determine \( S \) with coordinates \((F(x_0), 0)\) as the projection of \( R \) on the \( y \)-axis.
(iii) Project \( P \) horizontally to intersect the curve \( y = g(x) \) at \( Q \).
(iv) Project \( Q \) vertically to meet the line \( y = x \) at \( Q' \) and then extend \( Q' \) horizontally to intersect a vertical line through \( P \) at \( P' \).
(v) A line through \( P \) perpendicular to the line \( SP' \) gives the required direction (value of \( \frac{dx}{dy} \)) at point \( P \).

The proof is obvious from the figure on noting that the slope of line \( SP' \) is

\[ g(x_0)/(y_0 - F(x_0)). \]
More complicated versions of (5) and (6) can also be handled geometrically using the same ideas; however, the construction becomes correspondingly more complex, in general requiring one or more additional curves in the phase-plane diagram.

References