GENERATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS*

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In [1] Moseley derived some nonseparable solutions of the Helmholtz equation, and in [2] he demonstrated how these solutions could be generated by applying a certain differential operator to a separable solution. In the present paper we point out that Moseley's method applies in greater generality to any linear partial differential equation, and, indeed, to any linear operator equation.

Suppose L is a linear operator and that w is any solution to the equation

$$Lw = \lambda w. \tag{1}$$

Suppose, further, that the parameter λ is a function of two other parameters a and b: $\lambda = \lambda(a, b)$. Then the solutions of (1) depend on a and b. Assuming the operators L and $\partial/\partial a$ and L and $\partial/\partial b$ commute, we have from (1)

$$Lw_a = \lambda w_a + \lambda_a w \tag{2}$$

and

$$Lw_b = \lambda w_b + \lambda_b w. \tag{3}$$

Multiplying (2) by λ_b and (3) by λ_a and subtracting, we find

$$L(\lambda_b w_a - \lambda_a w_b) = \lambda(\lambda_b w_a - \lambda_a w_b) \tag{4}$$

which shows that the function

$$u = \lambda_b w_a - \lambda_a w_b \tag{5}$$

is a solution to $Lu = \lambda u$ if w is.

Moseley, in [2], considered the special case in which $L = \nabla^2$ and $\lambda = a^2 + b^2$. This enabled him to generate solutions of the Helmholtz equation by starting with the usual one obtained by separation of variables, and he carried this program out in both rectangular and polar coordinates. It is clear from the foregoing, however, that the restriction to special coordinate systems is inessential.

Although L can be any linear operator, the results may be trivial in some cases. For example, application of the foregoing method to the ordinary differential equation $y'' = \lambda y$ yields the solution $y \equiv 0$. However, for partial differential equations the method is quite fruitful, as [1] and [2] show.

Eqs. (2) and (3) also represent a sort of generalization of an idea of Brand [3], who pointed out that differentiating (1) with respect to λ yields a generalized eigenfunction. w_a is a generalized eigenfunction in the special case $\lambda = a$. So w_a and w_b are sorts of generalized eigenfunctions.

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Application to other types of equations. The method is not limited to equations of the form (1). The parameter λ can multiply terms other than the unknown function. For example, if $c = c(a, b)$, then the same method shows that

$$u = c_b u_a - c_a u_b \tag{6}$$

is a solution to

$$u_{xx} - c^{-2} u_{tt} = 0 \tag{7}$$

if u is. However, the generating solutions must be chosen judiciously. If a separated form for u, i.e. a standing wave form, is chosen, then Moseley's method will generate nonseparable solutions, but if the progressing wave form $u = f(x - ct)$ is chosen, the method deduces only the solution $u \equiv 0$.

The three-dimensional wave equation, $\nabla^2 u - c^{-2} u_{tt} = 0$, the telegraph equation, the heat equation, and many others have parameters in their coefficients that can be used as above to generate further solutions form known solutions.

Fractional operators. In [2], for the case $\lambda = a^2 + b^2$, Moseley uses integral powers of the operator

$$M = b\, \partial/\partial a - a\, \partial/\partial b \tag{8}$$

to obtain further solutions. Since M is essentially an angular derivative when a and b are interpreted as rectangular coordinates, we can use fractional powers of M, too. More precisely, if we introduce polar coordinates via the formulas $a = r \sin \theta$ and $b = r \cos \theta$, then we see that $M = \partial/\partial\theta$. Hence

$$M^n = \partial^n/\partial\theta^n = D_\theta^n \,. \tag{9}$$

Introducing the fractional, or Riemann–Liouville integral,

$$I_\theta^p f = \frac{1}{\Gamma(p)} \int_0^\theta (\theta - \phi)^{p-1} f(\phi)\, d\phi, \qquad p > 0, \tag{10}$$

and writing, for any $q > 0$, $q = n - p$, where $0 < p \leq 1$ and n is a nonnegative integer, we define, as usual,

$$D_\theta^q = D_\theta^n I_\theta^p \,. \tag{11}$$

Then $M^q = D_\theta^q$, with D_θ^q defined by (11), I_θ^p defined by (10) and $\theta = \arctan(a/b)$, when applied to solutions of $Lu = \lambda u$ or equations of the types mentioned above, generates further solutions of $Lu = \lambda u$, or those equations.

Again, it is not necessary to restrict ourselves to the special relation $\lambda = a^2 + b^2$. If $\lambda = \lambda(a, b)$, then the corresponding Moseley operator,

$$M_\lambda = \lambda_b\, \partial/\partial a - \lambda_a\, \partial/\partial b, \tag{12}$$

represents differentiation in the direction whose characteristic equations are

$$da/dt = \lambda_b \tag{13}$$

and

$$db/dt = -\lambda_a \,. \tag{14}$$

Combining (13) and (14), we find

$$\lambda_a(da/dt) + \lambda_b(db/dt) = 0, \tag{15}$$

or, $\lambda(a, b) = \text{const}$. Thus, if we put $\lambda(a, b) = \mu$ here, we see that, essentially, $M_\lambda = \partial/\partial\mu$. Introducing I_μ^p and D_μ^q as before, we can then define fractional powers of M_λ.

References

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[3] Louis Brand, *Matrices with specified eigenvalues and associated eigenvectors, proper or generalized*, Amer. Math. Monthly 74, 640–648 (1967)